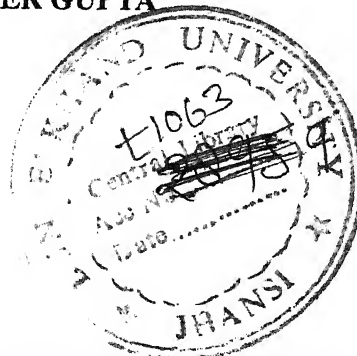


COST STRUCTURE IN SOME RELIABILITY MODELS

**THESIS SUBMITTED FOR THE AWARD OF
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN MATHEMATICS (OPERATIONAL RESEARCH)**

by

MAHESH CHANDER GUPTA



**DEPARTMENT OF MATHEMATICS AND STATISTICS
BUNDELKHAND UNIVERSITY, JHANSI (INDIA)**

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DECLARATION

The thesis entitled "COST STRUCTURE IN SOME RELIABILITY MODELS" is submitted by me in The Department of Mathematics and Statistics, Bundelkhand University, Jhansi for the award of the degree of Doctor of Philosophy is based on my original research work. This work either in part or full, has not been submitted to any other University or Institution for the award of any degree/diploma.



(MAHESH CHANDER GUPTA)

CERTIFICATE

This is to certify that the work embodied in the thesis "COST STRUCTURE IN SOME RELIABILITY MODELS" by Mahesh Chander Gupta, for the award of the degree of Doctor of Philosophy is a record of bonafide research work carried out by him under my supervision and guidance and has not been submitted elsewhere for a degree/diploma in any form.

It is further certified that he has worked with me for the period required under clause seven of the Bundelkhand University ordinance.



(Dr. V. K. Seghal)

READER

Department of Mathematics and Statistics
Bundelkhand University
Jhansi (INDIA)

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PREFACE

Reliability models generally deal with two types of problems depending upon system stages viz., design and operative stage. Problems in design stage pertain to redundancy allocation, reliability maximization and/or cost minimization etc., whereas problems in operative stage deal with analysis of stochastic behaviour of the systems and to determine optimal maintenance policies.

The review of existing literature on reliability reveals that authors who have contributed to the economic aspects of reliability had definite cost structures in their minds to superimpose on the system model. Generally speaking, reliability and cost are related parameters of a system and in fact other parameters of the system can be expressed in terms of cost.

It is needless to mention that problems at both these stages are important in their own right and require due attention. However, for the analysis of maintained systems, the problems at the latter stage (operative) play a dominant role because a system analyst loses his control over the design parameters once the system enters operative stage. So, for maintained systems there is a genuine demand of research investigation in the direction of repair/maintenance which can help a maintenance engineer/analyst in making an optimal choice of repair rule (for a fixed failure rate) and/or in having better understanding of system's economics by studying parameters' effectiveness from cost considerations.

The thesis consists of four chapters :

Chapter I : Introduction

Chapter II : Optimal Repair Rate in Some Maintained Systems

Chapter III : Optimal Failure Analysis in Series Systems

Chapter IV : Evaluation of Series System Through Dynamic Programming Technique.

Chapter I presents a critical review of the existing literature and a brief account of contribution made by the author.

Chapter II discusses the profit evaluation studies for some systems with repair maintenance. Three models describe single unit maintained systems under different operational conditions and one model deals with two-unit system with parallel redundancy. The concept of minor repair has also been applied in one of the models. The cost structure of Howard has been superimposed on the Markov process / semi-Markov Process generated by a system model. The expected profit thus obtained has been maximized and optimal repair rates are determined. This cost structure allows different cost rates (variables) in each state and different transition costs incurred at the time when the system makes a transition from one state to another. Following 4 models in this chapter are discussed in the following sections :

2.2 Profit evaluation in a single unit system

2.3 Profit evaluation in a single unit system with a less productive state.

2.4 Profit considerations in a single unit system with minor repair.

2.5 Profit evaluation in a 2-unit parallel redundant system.

Chapter III deals with failure analysis in series systems. Problems of failure classification and resource allocation on failure detection mechanisms have been considered. For the problem of failure detection in a series system, a cost structure involving probabilities of correct detection (depending upon cost spent on detection) has been mixed with Howard's reward structure and is superimposed on a 2-unit series system. Optimal allocation of cost on detection mechanisms has been determined when the total cost for detection is fixed. Further a problem of failure classification in a series system has been discussed after taking into account cost of misclassification whenever the system fails. A decision rule based on the life-time distribution of the system which minimizes the expected misclassification cost has been suggested. A procedure has also been suggested to examine the discriminatory power of the rule. There are following two formulations discussed in this chapter :

3.2 Optimal failure classification in an n -unit series system.

3.3 Optimal allocation of cost to detectors in a 2-unit series system.

Chapter IV discusses application of dynamic programming technique to solve problems in series systems. Problem of cost allocation has been formulated and solved through dynamic programming technique. This is discussed in the following section :

4.2 Optimum cost allocation in a series system under a constraint on the cost.

It may not be out of place to mention that the results contained in different chapters are based on seven research papers published in different international journals of repute by the author. It may be mentioned that the

procedure followed for identifying the equations in the thesis is to put numerals within parentheses, where the first figure refers to the chapter, the second represents the section number and the third represents the serial number of the equation, e.g., (3.2.5) denotes the 5th equation of section 2 of chapter III. The same pattern is followed to number the tables and figures also.

Abbreviations :

s-independent : statistically independent

↑ : monotonically increasing

↓ : monotonically decreasing.

CHAPTER I

INTRODUCTION

1.1 INTRODUCTION

The reliable performance of system is of immense interest in modern business, industrial and defence systems. In fact, increasing complexity of present day system has resulted in engineering problems involving high performance, reliability and maintainability. It is the importance of reliability in system analysis studies that has attracted attention of a large number of research workers from different disciplines to carry out their analysis from the reliability view point. Investigation of reliability models with cost considerations is of utmost importance to system analysts.

We develop below basic definitions and terminology which are needed to understand a reliability model.

1.2 Definitions and Concepts

The terms/parameters which are used in reliability may be divided into two categories :

(A) System configuration parameters

(B) System effectiveness parameters.

(A) System Configuration Parameters :

1. **Series system** :- is one in which a failure in any one of its units (components) leads to system failure.
2. **Redundant systems** :- are those where more than one means (units) are available for performing a function, so that when one means (unit) fails, others are available to perform the task.

3. **Parallel or active redundant systems** :- are those where all redundant units and basic units operate simultaneously rather than being switched on when needed.
4. **Standby redundant systems** :- are those in which only one unit operates at a time, while the other redundant units (components) are switched on sequentially only when needed i.e., upon the failure of the operating units. Here, the system is said to have failed when all the units have failed.

The standby redundancy may be cold, warm or hot.

(a) **Cold standby redundant systems** :- are those in which redundant units do not fail when they are in standby.

(b) **Warm standby redundant systems** :- are those where redundant units can fail when they are in standby. Normally, in such systems failure probability of standby unit is less than that of an operating unit.

(c) **Hot standby redundant systems** :- are those in which redundant units can fail when they are in standby with failure probability same as that of an operating unit.

5. **k-out-of-n : F system** :- it consists of n units and the system fails if and only if k ($< n$) out of its n units are failed. Such systems are prevalent in practical systems, e.g., a communication system with 4 transmitters in which the average message load is such that at least 3 transmitters must be operational for successful message delivery, thus the system is 2-out-of-4 : F system.

6. **k-out-of-n : G system** :- it is a redundant system composed of n units and the system functions if and only if at least k ($< n$) out of n units are operable. It is clear that series and parallel systems are n -out-of- n : G and 1-out-of- n : G systems respectively.
7. **Complex systems** :- are those systems which do not fall into the category of either standby or parallel or series-parallel systems.
8. **Imperfect switchover** :- The phenomenon which occurs when a standby unit is switched to operate at the time of failure of an operative unit in a redundant system is referred to as 'switchover'. Some times the switchover is not perfect i.e., not 100% error free and/or instantaneous. This is called 'imperfect switchover'.
9. **Maintainability** :- may be defined as a characteristic of design and installation which is expressed as the probability that a unit will be retained in or restored to a specified condition within a given period of time, when the maintenance is performed in accordance with certain rules. Maintenance includes all actions necessary for retaining a unit in or restoring it to a specified condition.

It is further divided into following two categories :-

(a) **Repair maintenance** :- whenever a unit fails, an activity is performed to restore the unit to a specified condition (operation).

(b) **Preventive Maintenance (P.M.)** :- this is performed in an attempt to retain a unit in a specified condition by providing systematic, inspection,

detection and prevention of failure.

P.M. is required for costly systems like computers, satellites, radar, transmitters etc., where cost of failure or replacement is quite heavy and systems are required to operate without failure for a long time span. There are several types of P.M. policies.

10. **Service replacements** :- are those replacement which are made after the occurrence of failure.
11. **Planned replacements** :- are those replacement which are made after the occurrence of failure. Such replacement may be made at scheduled times.
12. **Delayed repair** :- the repair of a unit is some times delayed due to any physical or environmental reason. This process is called delayed repair.
13. **Fault detector** :- occurrence of a fault (or failure) in a system is not always detected automatically. As such some mechanism may be used to detect the fault, this is referred to as fault detector.

(B) System effectiveness parameters :

1. **Reliability** :- may be defined as the probability that a system will perform its function adequately for a definite period of time under specified conditions.
2. **Pointwise availability** :- probability that the system will be able to operate within certain tolerances at a given instant of time.

3. **Interval reliability** :- probability that a system continues to operate during a given interval of time.
4. **Interval availability** :- expected proportion of a given interval of time that the system will be able to operate within certain tolerances.
5. **Steady-state availability** :- expected proportion of time that a system is operable when it is allowed to operate for a very long period.
6. **First passage time to system failure** :- this is defined as the probability that starting in a given state (up) system enters a failed state for the first time in time t .

That is

$$\varphi_{OF}(t) = P[\text{system goes to state 'F' in time } \leq t \mid \text{system is in 'O' at } t=0].$$

7. **Mean time to system failure** :- if T denotes time -to -first -system failure starting from the beginning, then MTSF is given by $E(T)$ where E stands for mathematical expectation.

This measure of system effectiveness appears to be more appropriate for the so called one-shot systems, e.g., missiles, rockets etc., which are used once for all.

Infact, first passage time distribution to system failure, $\varphi_0(t)$ and MTSF are related to the following manner.

$$\text{MTSF} = -\frac{d}{ds}\varphi_0^*(s) \Big|_{s=0}$$

where $\varphi_0^*(s)$ denotes L.S. transform of $\varphi_0(t)$.

8. **Expected profit** :- This is the total expected earnings per unit time of the system if the system is allowed to operate for sufficiently long period.

1.3 System Models in the Field of Reliability

Keeping in view the varied nature of practical problems, a large number of reliability models have been developed and analyzed in literature.

Most of the papers appeared are motivated to the following two directions:-

(1) **Analysis of special systems.**

(2) **Development of reliability, availability and cost optimization models.**

In the analysis of special systems one is interested in a specific system. The general approach for the analysis of such systems consists in defining and formulating the system model i.e., to state its assumptions, to enumerate different states and identify them as up and down ones, to develop the underlying equations governing the system behaviour and to apply an appropriate technique to obtain the solution.

For the second case viz., in the development of reliability, availability, cost optimization models, first of all one has to formulate the problem clearly i.e., to construct the objective function and constraints. Infact, in this area, formulation aspect is important in its own right, this involves clear understanding of effectiveness criterion, constraints visualisation, and their impact on objective functions. Once the problem is formulated successfully, next step is to search

and/or develop a suitable solution method to get optimal values for decision variables.

We now review the existing literature on reliability keeping in view the above classification.

(1) **Analysis of special system :-** In general following types of system models have been discussed under this category.

- (i) **Cold standby redundant systems :-** Muth (1966) considered a cold standby system with constant hazard rate for the on line unit where the repair time is a random variable having a gamma distribution and obtained expressions for the Laplace transform of mean time-to-system-failure (MTSF) of the system. Subsequently, Linton and Baraswell (1973) have generalized the results of Muth (1966) for case in which life time of the unit and its repair time are generally distributed random variables. Nakagawa and Osaki (1974a) obtained for the same system, expressions for the Laplace transform of the reliability, steady-state availability and its MTSF. Nakagawa and Osaki (1974b) derived expressions for the expected number of visits to a state. Srinivasan and Gopalan (1973a) obtained expressions for the Laplace transform of reliability and pointwise availability. Kumar (1977a) obtained expression for steady-state profit for a 2-unit cold standby redundant system.
- (ii) **Warm standby redundant systems :-** Branson and Shah (1971) considered a 2-unit system with exponential failure times and

general repair times and obtained the MTSF and steady-state availability. Srinivasan and Gopalan (1973b) obtained the Laplace transform of pointwise availability of the system comprising of dissimilar units by employing supplementary variable technique. Kumar (1977d) considered a 2-unit system with exponential failure times and general repair times and obtained the expression for expected profit of the system. Birolini (1975) analyzed his model when the failure time distribution of the on line unit follows general distribution and failure time of the standby unit follows exponential distribution, repair times are also of exponential nature by employing regenerative stochastic processes.

- (iii) **Parallel redundant systems :-** Gaver (1963) studied a 2-unit parallel redundant system with constant state dependent hazard rate and arbitrary repair and obtained the Laplace transform of reliability of the system its MTSF and steady-state unavailability, using supplement variable technique. Kulshrestha (1968, 1970), Mine and Kawai (1974), Linton and Saw (1974) have studied some parallel redundant systems. Kodama and Deguchi (1974) obtained the MTSF of a parallel redundant system with Erlangian failure and general repair. Linton (1976) extended these results for the case when the failure and repair distributions of one of the units are Erlangian and those of the other are arbitrary by employing supplementary variable technique.

- (iv) **2-Unit standby redundant system with imperfect switch over :-**

The importance of switch in standby redundant systems has attracted the attention of many research workers to incorporate switch behaviour in investigating the stochastic behaviour of standby redundant systems. Gopalan (1975), Osaki (1972a) Prakash (1973), Nakagawa and Osaki (1975), Nielsen and Runge (1974), Kumar (1977b) have studied such systems under different operational circumstances. Chow (1972, 1973) constructed mathematical models for the reliability of modularly redundant systems with unequal failure rates for the operating and standby units and different types of switch failures. Arnold (1973) introduced the concept of 'coverage' for repairable systems.

- (v) **Redundant systems with delayed repair :-** When a unit or switch fails; repair facility may not be available because of its preoccupation with higher priority jobs and as such failed unit might have to wait for sometime. Several other reasons may also be attributed to delay in repair. The reliability models with delay in repair have been analyzed by a large number of workers. [Gopalan and Saxena (1977), Kapur and Kapoor (1975a), Khalil and Bogus (1975), Khalil (1977b), Kumar and Lal (1979) etc.].
- (vi) **Redundant systems with two types of repair :-** Some electrical and mechanical devices can fail in two or more mutually exclusive failure modes. A vending machine or public telephone fails due to failure of units under benign environments or to the failure of human misuse. The failure (and thus equivalently repair) may

be categorized in 2 classes. Many papers with 2 types of repair have appeared in the literature, [Garg and Kumar (1977), Khalil (1977a,b), Kumar and Agarwal (1978), Osaki and Okumoto (1977), Proctor and Singh (1976) etc.].

- (vii) **Complex systems** :- All redundant systems may not be either standby or parallel or series-parallel. Consider a complex system consisting of two classes of components denoted by L_1 and L_2 . Class L_1 has n components in series and class L_2 has m identical components in parallel redundancy. Such systems have been investigated by Garg (1963a,b), Gupta (1973a,b), Sharma (1974), Agarwal (1975). Such systems have also been considered by Varma (1973) with repair priority and by Kapur and Kapoor (1975b) with demand pattern.

Different techniques were used by research workers in the evaluation study of reliability models to obtain parameters of interest. For example Gnedenko et al. (1965), Osaki (1970b) and Buzacott (1971) have used the renewal theoretic approach for the analysis of their models. The technique of Markov-renewal process has been successfully employed for the analysis of redundant systems by Srinivasan (1968), Osaki (1970a, 1972a,b), Osaki and Okumoto (1977). Further, Nakagawa and Osaki (1974a) have suggested a unique modification of MRP which is useful for the analysis of 2-unit redundant systems under general failure and repair time distributions. Later, this technique was employed in a large number of papers. [Nakagawa and Osaki (1974b, 1975), Kapur and Kapoor (1975a,b), Kapil and Sinha (1981)]. Gnedenko (1967),

Srinivasan et al. (1973) have used regeneration point technique for the analysis of redundant systems. The supplementary variable technique has been used in standby redundant system by Gaver (1963) and subsequently by others, Garg (1963a,b), Garg and Kumar (1977) to treat general distributions for repair times. Various other methods have also been applied by workers to analyze their models, e.g., discrete transforms Kulshrestha (1972), conditional transforms, Linton and Braswell (1973).

Branson and Shah (1971) applied the results from the theory of semi-Markov processes to reliability analysis for a 2-unit warm standby systems. Later on, this was applied to other systems [Kumar (1976, 1977d), Kumar and Lal (1979,1980)].

But a close look at the processes considered in these papers reveals the fact that the underlying process is not SMP in true sense. However, it looks so. The interesting fact is that the process satisfies certain conditions [Arndt (1977), Arndt and Franken (1977)] under which results from SMP theory are applicable. This has been pointed out in Kumar et al. (1981).

(2) Development of Reliability, Availability and Cost Optimization

Models :- A large number of developments have appeared to optimize system effectiveness: reliability, availability, cost, profit etc. For problems of this nature, Tillman et al. (1980) may serve as a good reference. We review below some of the important optimization models.

- (i) **Optimization models in series and parallel system :-** In these models there are several units (stages) say, connected in series and the units are statistically independent. If reliability of the

j^{th} unit is denoted by R_j , the system reliability R is given by

$$R = \prod_{j=1}^n R_j$$

Many workers [Ghare and Taylor (1969), Beraha and Misra (1974)] have made attempts to develop algorithms which maximize R subject to k resource constraints of the form

$$\sum_{j=1}^n h_{ij}(R_j) \leq \alpha_i \quad i = 1, 2, \dots, k$$

where $h_{ij}(R_j)$ denotes the requirement of the i^{th} resource and α_i stands for availability of the i^{th} resource. The functions $h_{ij}(R_j)$ may be linear or nonlinear depending upon situations under investigation.

Another useful area in series optimization models is redundancy allocation in an optimal manner. Banerjee and Rajamani (1973), Shetty and Sengupta (1975), Luus (1975) have considered problems to choose the optimum number of components, say m_j in parallel at each of the n stages in a series system so that reliability is maximized. Mathematically, the problem in general form is

$$\text{Max } R ; R = \prod_{j=1}^n (1 - Q_j^{m_j}), \quad Q_j = 1 - R_j$$

subject to the constraints given earlier.

Misra and Ljubojevic (1973), Tillman et al. (1977b) considered a simultaneous reliability and redundancy allocation problem viz.,

$$\text{Max } R ; R = \prod_{j=1}^n [1 - (1 - R_j)^{m_j}]$$

subject to

$$\sum_{j=1}^n h_{ij}(R_j) = \sum_{j=1}^n h_{ij}(R_j, m_j) \leq \alpha_i$$

and choose R_j and m_j .

They have used Lagrangian Multiplier technique to solve the problem. Shetty and Sengupta (1975) applied slacked sequential unconstraint minimization technique (SLUMT) which is superior to SUMT used by Tillman et al. (1970) for solving their problems.

Another formulation in series models has been due to Aggarwal (1977). He discussed the problem of minimizing the total cost under a lower bound on system reliability. In other words,

$$\text{Min } C ; C = \sum_{j=1}^n C_j(m_j)$$

subject to

$$R = \prod_{j=1}^n R_j(m_j) \geq R_0$$

$C_j(m_j)$ being the cost of the j^{th} stage when m_j parallel components are used at this stage. He used heuristic approach for the solution.

Misra (1971a) gave a dynamic programming formulation of redundancy allocation at each stage under two nonlinear constraints.

The problem of choosing MTBF, MTTR and m_j by cost minimization subject to availability constraints has been formulated by Lambert et al. (1971). They have taken total cost C and availability A for a parallel series system

as

$$C = \sum_{j=1}^n (R_j, M_j, m_j)$$

$$A = \prod_{j=1}^n A_j(R_j, M_j, m_j)$$

$$A_j = \frac{R_j}{R_j + M_j}$$

R_j and M_j are MTBF and MTTR at the j^{th} stage respectively.

The problem is to minimize C by choosing R_j, M_j, m_j within permissible design requirements so as to meet a specified level of availability.

A typical design requirement for the j^{th} stage is

$$R_j \geq 4m_j, \quad M_j \leq 5$$

Dynamic programming technique was used to obtain the solution.

- (ii) **Reliability optimization models with repair schemes :-** Pal and Bhattacharjee (1978) considered a system having n stages and j^{th} stage has m_j units. The objective is to maximize system reliability subject to nonlinear constraints depending upon maintenance. The system reliability is

$$R(t) = \prod_{j=1}^n R_j(t) = \exp(-Zt)$$

where

$$Z = \sum_{j=1}^n Z_j$$

and

$$Z_j = \frac{[(1-\rho_j)^2 \rho_j^{m_j-1} \lambda_j]}{1-\rho_j^{m_j} [1+m_j(1-\rho_j)]}$$

$$\rho_j = \frac{\text{failure rate of stage } j}{\text{repair rate of stage } j} = \frac{\lambda_j}{\mu_j}$$

The steady-state probability P_{0j} of stage j with its units being idle is

$$P_{0j} = \frac{1 - \rho_j}{1 - \rho_j^{m_j}}$$

The problem is

$$\text{Max}_{m_j} R(t)$$

subject to

$$m_j \geq 1 \quad (j = 1, 2, \dots, n)$$

$$\sum_{j=1}^n h_{ij} m_j \leq C_j \text{ (resource constraints)}$$

$$t(1-P_{0j}) \leq T_j^* \text{ (busy period constraints)}$$

(iii) **Optimization in repair, replacement and inspection models :-**

Replacement is carried out when the failures cause serious damages and/or imply expensive repairs. Replacement can be made in accordance with a predefined policy.

Replacement theory is of great importance in reliability theory and has been investigated by many research workers. Barlow and Proschan (1965) discussed several interesting replacement problems and obtained the optimum

repair policies minimizing the expected cost per unit time. Glasser (1967) considered the age replacement problem of Barlow and Proschan (1965) and gave graphs of the optimum repair policies. Fox (1966) introduced discount rate for a replacement problem. Scheaffer (1971) investigated a replacement model with an increasing cost for an operating unit. Nakagawa and Osaki (1974c) obtained the optimum policies for a replacement model with delay. Nakagawa and Osaki (1974d) have taken a repair limit replacement policy which is defined as the policy in which the repair is stopped if it is not completed within a fixed time (called the repair limit time). The problem is to determine the optimum repair time limit τ , which minimizes the expected cost per unit time for an infinite time span. Kapil and Sinha (1981) have discussed repair limit suspension policy for a 2-unit redundant system with 2-phase repair. A failed unit is first repaired by a phase 1 repair.

If the repair is not completed in a fixed time then the unit goes for a phase 2 repair. They have obtained optimum repair limit suspension policies which maximize the availability using MRP results.

Muth (1977) also studied a similar problem in which a replacement takes place at the first failure time after a predetermined age τ .

Mine and Kawai (1975) discussed an optimal inspection and replacement policy for a unit which assumes any one of several Markov states. The policy evaluation function is expected cost per unit time over an infinite time span. The problem is formulated as a SMDP with a modified policy improvement routine.

Alam and Sarma (1974) considered maintenance policies for a machine

with degradation in performance with age and subject to failure.

Hastings (1969) developed a repair limit replacement method. In this, when an item requires repair it is first inspected and the repair cost is estimated. If the estimated cost exceeds a certain amount known as the repair limit, then the item is not repaired but replaced. Two types of problems, one in which condition of equipment is related to age and other in which condition is related to number of major repairs an item has had, have been analyzed. Dynamic Programming methods have been used to obtain optimum repair limits.

Diveroli (1974) has considered simultaneously the problems of preventive replacement and service replacement. Several procedures have been developed to determine optimal policies. He also proves the optimal preventive replacement policy is a function of the optimal repair limit.

- (iv) **Profit optimization in redundant system models :-** Expected steady-state profit in redundant systems has been obtained in several papers [Kumar (1976), Kumar and Lal (1979)]. This measure of system performance was used to determine optimal P.M. in Kumar (1976). However, profit optimization problem was considered in Kumar and Lal (1980) to determine optimal maintenance policy for a redundant system. This method is based on Howard's (1971) policy iteration. Srinivasan et al. (1971) have considered expected profit in a 2-unit redundant system by mixing of two renewal processes. Further, Kumar and Kapoor (1981a) extended the solution method to find optimal maintenance policy

for a redundant system when the cost structure is discounted one.

- (v) **Optimization models with fault detection :-** The concept of fault detection in series system is important and needs attention. Gross (1970) had studied a 2-unit series system with fault detector, misclassification cost. He had minimized the cost of misclassification and determined the optimal cost spent on detector. Later, Kumar (1975) had extended the result to the case of n-unit series system. He obtained a sufficient condition under which a detection mechanism would be economically feasible i.e., expected cost of misclassification is minimized. Recently, Takami et al. (1978) have developed a Markov model for a class of series systems which have fault detectors to find component failures. The optimal allocation of fault detectors is determined by formulating the problem as 0-1 integer programming problem.

Recently, Kumar and Kapoor (1981b) have solved a problem concerning classification of failure in a 2-unit series system, the classification criterion depends upon the time to system failure.

1.4 Some Cost Structures in Reliability Models

The review of existing literature in previous sections reveals that authors who have contributed to the economic aspects of reliability had definite cost structures in their minds to superimpose on the system model. Generally speaking, reliability and cost are related parameters of a system. But a unique

expression of this relationship which is valid in reliability modelling is not feasible mathematically. Never the less, some properties originating from the conceptual frameworks which are desired of cost functions are :-

- (I) Components with very low reliability cost very low.
- (II) Similarly, components which are highly reliable cost very high.
- (III) Cost is an increasing function of reliability.
- (IV) Derivative of cost with respect to reliability is an increasing function of reliability.

We now discuss some important cost structures which have appeared in reliability modelling and serve a useful purpose to system analysts :-

(1) Beriphol (1961) has expressed the cost of a component as a function of its reliability as given below

$$C_n = \frac{K_{1n}}{1-R_n(C_n)} \exp [-K_{2n}(1-R_n(C_n))]$$

where

C_n : Cost of the n^{th} component

$R_n(C_n)$: reliability of the n^{th} component

K_{1n}, K_{2n} : constants depending upon components.

A modified version of the above cost structure has appeared in Aggarwal and Gupta (1975)

$$C_n = K_n \left[\tan\left(\frac{\pi}{2}R_n\right) \right]^{f(R_n)}$$

here $f(R_n)$ is a function of R_n . In many situations $f(R_n)$ takes the form

$$f(R_n) = 1 + R_n^{x_n}, \quad 0 < x_n < 1$$

$$f(R_n) = m_n, \quad 0 \leq m_n \leq 2$$

(2) In the problem of maximizing R defined on page 12

$$R = \prod_{j=1}^n [1 - (1 - R_j)^{m_j}]$$

subject to

$$\sum_{j=1}^n h_{ij}(R_j) \leq \alpha_i$$

and to choose m_j and R_j simultaneously, Misra and Ljubojevic (1973) have taken the functions $h_{ij}(R_j)$'s as

$$h_{ij}(R_j, m_j) = a_{ij} \exp[-b_{ij}/(1 - R_j)]$$

In fact most of the models assume additivity of costs i.e., total cost C is additive in terms of the costs of the constituent units

$$C = \sum_{j=1}^n C_j(m_j)$$

where $C_j(m_j)$ is the cost of the j^{th} stage assumed to be a known function of m_j .

(3) Many developments are based on expected cost per unit time for an infinite time span. Nakagawa and Osaki (1974d), Kontoleon (1977), Muth (1977), Mine and Kawai (1974, 1975) have used this concept in their works. This is obtained as follows.

Find the expected cost from the instant a system starts operating upto the completion of the repair/replacement. (This constitutes one complete cycle). Divide this quantity by the mean time of one cycle.

For example, if $F(t)$ is cdf of repair time of a system, C_0 is the replacement cost, $C_r(t)$ is the expected repair cost incurred during $(0, T]$ then the expected total cost over one cycle is

$$\begin{aligned} C &= [C_0 + C_r(\tau)] \bar{F}(\tau) + \int_0^{\tau} C_r(t) dF(t) \\ &= C_0 \bar{F}(\tau) + \int_0^{\tau} \bar{F}(t) dC_r(t) \end{aligned}$$

where τ is repair limit time.

The mean time of the cycle is

$$\lambda + \tau \bar{F}(\tau) + \int_0^{\tau} t dF(t) = \lambda + \int_0^{\tau} \bar{F}(t) dt$$

Thus the expected cost per unit time over an infinite time span is

$$L(\tau) = \frac{C_0 \bar{F}(\tau) + \int_0^{\tau} \bar{F}(t) dC_r(t)}{\lambda + \int_0^{\tau} \bar{F}(t) dt}$$

For $C_r(t)$ the two usual forms are :

$$C_r(t) = at^b, \quad a > 0, \quad 0 \leq b < 2$$

i.e., repair cost is proportional to time and

$$C_r(t) = a(e^{bt} - 1), \quad a > 0, \quad b > 0$$

i.e., exponential repair cost.

The problem is to obtain optimal τ which minimizes $L(\tau)$

(4) Shershin (1970) has discussed a problem of simultaneous apportionment of reliability and maintainability. He visualized the total cost function C_N for a system with N subsystems as the sum of the failure rate and maintainability costs. Assuming exponential failure rates, the failure rate λ_s of the system and the failure rate for the k^{th} subsystem are related by

$$\lambda_s = \sum_{k=1}^N \lambda_k$$

Further, let $d_k(\lambda_k)$ be the cost for achieving failure rate λ_k for the k^{th} subsystem. Then, the cost of achieving a system failure rate λ_s is

$$C_N(\lambda_s) = \sum_{k=1}^N d_k(\lambda_k)$$

He has concentrated on maintenance time and has considered 3 major activities and their associated costs to be performed under maintenance time.

- (i) Maintainability during design stage;
- (ii) Performing maintenance to prevent failures termed as P.M.; and
- (iii) Repairing a failed subsystem/subsystems termed as C.M. associated with each component.

(5) Nakagawa and Osaki (1976a) considered a problem of optimization for a one-unit reliability model which is provided with n spares. They assumed in the model that whenever, the main unit fails, it undergoes repair and the spare unit starts functioning. After the repair of the main unit, it begins operation whereas the spare unit acts as standby. However, if the spare unit

fails before the repair of the main unit, it is not repaired and it is scrapped, the other spare unit starts operation. In this way, the first failure occurs when the main unit is under repair and the last spare unit fails. In the analysis of results to obtain total expected cost $C(t)$ in $(0,t]$, they used following costs :

C_r : cost of repairing the failed unit

C_s : cost of replacing spare unit.

Then, C , the expected cost per unit time for an infinite time span is

$$C = \lim_{t \rightarrow \infty} \frac{C(t)}{t} = C_r \cdot [\text{expected number of failed units per unit time}] + C_s \cdot [\text{expected number of spares per unit time}].$$

(6) Okumoto and Osaki (1977) discussed a cost structure in the context of obtaining preventive maintenance policies for a standby system. They assumed in their model that preventive maintenance is allowed for an operating unit only and when it is supported by a standby unit. But the failed unit is replaced as it cannot be repaired.

They have considered the following costs in their model.

C : replacement cost of a failed unit

a : preventive maintenance cost per unit time.

Then, the expected cost $C(t)$ per unit time according to them is given by

$$C(t) = a \cdot [\text{the limiting probability that a unit is under preventive maintenance}] + C \cdot [\text{expected number of replacements per unit time}].$$

(7) Cleroux and Hanscom (1974) have taken a cost structure which takes into account adjustment costs, depreciation costs or interest charges which are paid at fixed equidistant intervals of time. They minimized the average cost per unit time over an infinite time span to determine optimal age replacement policy for a single unit system.

In notation let

C_1 : cost for each replacement of a failed unit

C_2 : cost for replacement of a non failed unit

$C_3(ik)$: cost which is incurred at the age ik ; $i=1, 2, \dots$

Above costs are known. Further, the sequence of costs $[C_3(ik)]$ helps in keeping the unit operative. The cost $[C_3(ik)]$ may include adjustment costs, depreciation costs, interest charges etc.

(8) In the problems related to detection of failures as and when they occur, some interesting cost structures have appeared in the literature Kumar (1975), Gross (1970), Kumar and Kapoor (1981b), Takami et al. (1978)].

Kumar (1975) considered the problem of detection of failures in an n -component series system. He obtained a condition under which detection mechanism is economically feasible using the following cost structure

$p_{ij}(c)$: Prob. [detecting the j^{th} component as failed whereas infact i^{th} component has failed]

where c is the amount of money spent on the detection mechanism that enhances the probability of correctly detecting the failed component.

d_{ij} : extra incurred cost if i^{th} component has failed but failure is assigned to j^{th} component erroneously.

These $p_{ij}(c)$ satisfy the following properties.

$$\sum_{j=1}^n p_{ij}(c) = 1 \text{ for } i = 1, 2, \dots, n$$

$$0 \leq p_{ij}(c) \leq 1 \text{ for } i, j = 1, 2, \dots, n$$

$$p_{ij}(0) = \frac{1}{n} \text{ for } i, j = 1, 2, \dots, n$$

$$p_{ij}(c) \geq \frac{1}{n} \text{ for } i = j = 1, 2, \dots, n \text{ and } c \neq 0$$

$$p_{ij}(c) \leq \frac{1}{n} \text{ for } i \neq j = 1, 2, \dots, n \text{ and } c \neq 0$$

$$p_{ii}(c) \uparrow c \text{ for } i = 1, 2, \dots, n$$

$$p_{ij}(c) \downarrow c \text{ for } i \neq j = 1, 2, \dots, n$$

$$\lim_{c \rightarrow \infty} p_{ij}(c) = \delta_{ij}$$

where δ_{ij} is Kronecker's delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Gross (1970) is a particular case of the above model when the number of components is 2.

(9) Kumar and Kapoor (1981b) have developed a classification rule for a 2-unit series system based on the life-time distribution of the system which minimizes the expected cost due to misclassification.

They have considered the following costs.

c_{12} : cost of classifying the unit 2 as failed whereas unit 1 has failed.

c_{21} : cost of classifying the unit 1 as failed whereas unit 2 has failed.

(10) Takami et al. (1978) have obtained the solution for the problem of optimal allocation of fault detectors in a series system. They have considered following costs for their model.

c_i : cost of the detector allocated to component i

c_s : loss per unit time caused by the system failure.

(11) The following reward (cost) structure due to Howard (1971) has been extensively applied to obtain expected profit in redundant systems operating under different operational conditions, Kumar (1976, 1977c).

The cost structure is :-

Y_i : earning rate of the system while it is in the state S_i

r_{ij} : fixed transition cost incurred at the time of transition from S_i to S_j .

It may be remarked that the above cost structure allows both types of costs viz., fixed and variable costs.

1.5 SUMMARY OF THE THESIS

Contribution and Distribution of Chapters :- The review given in the preceding pages reveals that for non-maintained systems problem of redundancy allocation is important whereas for efficient running of maintained systems it is worthwhile to determine optimal failure/repair rates. If a system is in the

working state (i.e., it is already designed) one will be interested in having a control over the repair process because failure properties might have already been prespecified in design stage. So, for maintained systems there is a genuine demand of research investigations in the direction of repair/maintenance which can help a maintenance engineer/analyst in making an optimal choice of repair rule (for a fixed failure rate) and/or in having better understanding of system's economics by studying parameters' effectiveness from cost considerations.

Keeping the above in view, the efforts in the thesis are motivated in the following directions.

- (i) To provide a decision maker with readily available values of repair rates depending upon the earning rates and failure characteristics of the system which optimizes steady-state expected profit of the system and to provide basic cost equations for the better understanding of system's economic behaviour.
- (ii) To suggest some guidelines to the management regarding installation of a new system/or replacing the existing one with a new one whose parameters like failure rate, repair rate, earnings in various states (viz., operable and failed) are known.
- (iii) Further, in maintained systems, it is of great importance to analyze failures i.e., to locate/detect a failure, to classify a failure, to isolate the faulty component/components. In this direction, two problems of failure detection and classification have also been formulated and solved.
- (iv) In the context of the non-maintained systems, a reliability

maximization problem in an n -component series system under constraint on cost has been discussed.

While formulating the problems at (i) and (ii) and discussing the solution procedures, the cost structure of Howard (1971) has been superimposed on the Markov process/semi-Markov process generated by a system model. The expected profit thus obtained has been maximized and optimal repair rates are determined. This cost structure allows different cost rates (variables) in each state and different transition costs incurred at the time when the system makes a transition from one state to another.

For the problem of failure detection in a series system, a cost structure involving probabilities of correct detection (depending upon cost spent on detection) has been mixed with Howard's reward structure (1971) and is superimposed on a 2-unit series system. Optimal allocation of cost on detection mechanisms has been determined when the total cost for detection is fixed. Further, a problem of failure classification in a series system has been discussed after taking into account costs of misclassification whenever the system fails. A decision rule based on the life-time distribution of the system which minimizes the expected misclassification cost has been suggested. A procedure has also been suggested to examine the discriminatory power of the rule.

For the problem of cost allocation to different components in a series system under a constraint on the total cost, a useful and interesting reliability cost function is devised and a solution procedure is developed based on dynamic programming.

The thesis consists of four chapters :

- Chapter I : Introduction
- Chapter II : Optimal Repair Rate in Some Maintained Systems
- Chapter III : Optimal Failure Analysis in Series Systems
- Chapter IV : Evaluation of Series Systems Through Dynamic Programming Technique.

Chapter II discusses the profit evaluation studies for some systems with repair maintenance. Three models describe single unit maintained systems under different operational conditions and one model deals with 2-unit system with parallel redundancy. The concept of minor repair has also been applied in one of the models. Optimal repair rates maximizing expected profit have been obtained analytically for almost all models. The four models on profit evaluation studies in this chapter are discussed in the following sections :

- 2.2- Profit evaluation in a single unit system
- 2.3- Profit evaluation in a single unit system with a less productive state
- 2.4- Profit considerations in a single unit system with minor repair
- 2.5- Profit evaluation in a 2-unit parallel redundant system.

Chapter III deals with failure analysis in series systems. Problems of failure classification and resource allocation on failure detection mechanisms have been considered in the following two sections :

- 3.2- Optimal failure classification in an n-unit series system.

3.3- Optimal allocation of cost to detectors in a 2-unit series system.

The last chapter discusses application of dynamic programming technique to solve problem in series systems. Problem of cost allocation has been formulated and solved through dynamic programming technique. This is discussed in the following section :

4.2- Optimum cost allocation in a series system under a constraint on the cost.

CHAPTER II

OPTIMAL REPAIR RATE IN SOME MAINTAINED SYSTEMS

2.1 INTRODUCTION

Maintained systems are quite prevalent and useful in modern defence/industrial systems. During last three decades a large number of papers have appeared to analyse such systems under different conditions. These generalizations (contributions) are motivated to the following objectives :

- (i) To generalize the underlying distributions. These distributions may be for failure time, repair-time and/or for other features of the system units [Nakagawa and Osaki (1974a), Srinivasan and Gopalan (1973a), Garg (1963b)] etc.
- (ii) To incorporate certain new parameters in existing models. These new parameters may be of interest to system designers and maintenance engineers [Kumar and Lal (1979), Arnold (1973), Chow (1975), Gopalan and Saxena (1977), Khalil and Bougas (1975), Kapoor and Kapur (1975a), Kumar and Jain (1977)] etc.
- (iii) To obtain and analyse new system parameters which may serve useful purpose while making evaluation studies [Kumar (1976), Mine and Kawai (1974)] etc.

Garg (1963b) has taken a general distribution for repair-time and analysed a redundant system model. For quite a long time, workers have been assuming at least one of the two distributions : failure-time and repair-time as exponential. Srinivasan and Gopalan (1973a) have analysed a 2-unit system model assuming both the distributions as general. They obtained reliability and availability using regenerative properties. Later, Nakagawa and Osaki (1974a) developed a unique modification of MRP and discussed in detail

reliability characteristics of a 2-unit cold standby system.

As regards inclusion of new parameters of interest to system designers/analysts, some of the interesting parameters include 'delayed repair' Khalil and Bougas (1975), Kapoor and Kapur (1975a), Gopalan and Saxena (1977), Kumar and Jain (1977) : 'Intermittent repair' and 'contact failure' Kumar and Lal (1979), 'switch failure modes' Nakagawa (1977a), 'automatic recovery' Arnold (1973) etc.

Further, Kumar (1976) obtained expected profit for 2-unit systems operating under different conditions. This parameter has been suggested as the measure of system effectiveness and optimal preventive maintenance policies are also discussed. Mine and Kawai (1974) considered a 2-unit parallel system with good, degraded and failed states and discussed a preventive maintenance policy that maximizes the expected profit rate of the system over an infinite time span. Mine and Kawai (1975) discussed optimal replacement policy for a unit which assumes any one of several Markov states. The problem was formulated as a Markov decision process with a modified Policy Improvement Routine and evaluation function is the expected cost per unit time over an infinite time span.

Above review reveals that mostly the contributions are concerned with determination of system parameters and/or inclusion of new features of units constituting the system. Nevertheless, efforts have been in the direction of helping system analysts and decision makers to take optimal decisions as and when they are needed. These papers generally provide answer to the following broad question :

What should be the optimal preventive maintenance policy to maintain the system so that the overall system performance (availability, reliability, cost, profit etc.) is optimised?

The purpose of the present chapter is two fold :

- (a) To develop cost equations for a maintained system and obtain thereby expected profit of the system in a given time interval.
- (b) To obtain optimal values for repair rates which may help in better management of a maintained system.

It may be worthwhile to point out here that in all the papers Kumar (1976, 1977a, b, c, d) derived expected profit by direct substitution of system parameters in Howard's (1971) expression for g (expected profit). Whereas we write basic equations for two models to derive the expression for g . These equations help in better understanding of the system's economic behaviour and provide better insight into solution method. However, the result for the other two models have been derived by using Howard's (1971) technique.

We discuss following models in the subsequent sections :

Model 1 :- Profit evaluation in a single unit system

Model 2 :- Profit evaluation in a single unit system with a less productive state

Model 3 :- Profit considerations in a single unit system with minor repair

Model 4 :- Profit evaluation in a 2-unit parallel redundant system.

2.2 Model 1 :- PROFIT EVALUATION IN A SINGLE UNIT SYSTEM

System Model

- (i) There is a 1-unit system.
- (ii) Failure-time distribution of the system is exponential with parameter β .
- (iii) Whenever the system fails, it undergoes repair.
- (iv) Repair-time distribution of the system is exponential with parameter μ .
- (v) Repair facility always exists.
- (vi) After the repair, the system acts as new one.
- (vii) All probability distributions are s-independent.
- (viii) The system earns at a fixed rate so long as it is operable.
- (ix) The system loses at a fixed rate for the time it is under repair.
- (x) There is a fixed transition reward (cost) whenever the system changes its state viz., operable (failed) to failed (operable).

System states and transitions

The system can be in one of the following 2 states :

0 : operating

1 : failed

To identify the system at any instant, let

S_0 : system is operating

S_1 : system is in failed state and is under repair.

Notation

β : constant failure rate of the system

μ : constant repair rate of the system

r_{ii} : cost per unit time or earning rate in S_i ($i=0, 1$)

r_{ij} : transition cost for a transition from state S_i to S_j , $i \neq j$
($i, j=0, 1$)

$V_i(t)$: total expected earning of the system for a future period of length 't' given that at $t=0$ the system was in S_i ($i=0, 1$)

Δ : small interval

$\bar{f}(s)$: Laplace transform of $f(t)$, i.e.,

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \text{Re}(s) > 0.$$

Cost Equations and System Profit

Following Howard (1960), the following cost equations can be written

$$V_0(t+\Delta) = [r_{00}\Delta + V_0(t)][1-\beta\Delta] + [r_{01} + V_1(t)]\beta\Delta \quad (2.2.1)$$

$$V_1(t+\Delta) = [r_{11}\Delta + V_1(t)][1-\mu\Delta] + [r_{10} + V_0(t)]\mu\Delta \quad (2.2.2)$$

Initially $V_i(0) = 0$ ($i = 0, 1$).

The equation 2.2.1 is interpreted in the following manner.

During the short time interval Δ , the system which started in '0' state at $t=0$ may remain in '0' state or make a transition to state '1'. If it remains

in '0' state for time Δ , the system's earning will be $r_{00} \Delta$ plus the expected earning it will make in the remaining time of 't' units viz., $V_0(t)$. The probability that it remains in state '0' for time Δ is 1 minus the probability that it makes a transition to state '1' in Δ , i.e., $1-\beta\Delta$. In case the system makes a transition to state '1' during time interval Δ , the system's earning will be r_{01} plus the expected earning it will make when the system starts in state '1' with time t remaining viz., $V_1(t)$. The sum of the products of earnings with their respective probabilities gives the total expected earning. Equation 2.2.2 can be interpreted in the same manner.

Expanding equations 2.2.1 and 2.2.2 we get

$$V_0(t+\Delta) = r_{00}\Delta + V_0(t) - \beta V_0(t)\Delta + \beta r_{01}\Delta + \beta V_1(t)\Delta + 0(\Delta) \quad (2.2.3)$$

$$V_1(t+\Delta) = r_{11}\Delta + V_1(t) - \mu V_1(t)\Delta + \mu r_{10}\Delta + \mu V_0(t)\Delta + 0(\Delta) \quad (2.2.4)$$

Neglecting terms of higher order of Δ and letting $\Delta \rightarrow 0$, we obtain the following system of differential equations

$$\left[\frac{d}{dt} + \beta \right] V_0(t) = r_{00} + \beta r_{01} + \beta V_1(t) \quad (2.2.5)$$

$$\left[\frac{d}{dt} + \mu \right] V_1(t) = r_{11} + \mu r_{10} + \mu V_0(t) \quad (2.2.6)$$

Applying Laplace transforms to the set of equations 2.2.5 and 2.2.6 and employing initial conditions $V_i(0)=0$, ($i=0, 1$) we get

$$\bar{V}_0(s) = \frac{r_{00} + \beta r_{01}}{s(s+\beta)} + \frac{\beta}{s+\beta} \bar{V}_1(s) \quad (2.2.7)$$

$$\bar{V}_1(s) = \frac{r_{11} + \mu r_{10}}{s(s+\mu)} + \frac{\mu}{s+\mu} \bar{V}_0(s) \quad (2.2.8)$$

On substituting the value of $\bar{V}_1(s)$ from (2.2.8) in (2.2.7), we get

$$\bar{V}_0(s) = \frac{(r_{00} + \beta r_{01})s + \mu(r_{00} + \beta r_{01}) + \beta(r_{11} + \mu r_{10})}{s^2(s + \beta + \mu)} \quad (2.2.9)$$

On substituting the value of $\bar{V}_0(s)$ from (2.2.7) in (2.2.8), we get

$$\bar{V}_1(s) = \frac{(r_{11} + \mu r_{10})s + \beta(r_{11} + \mu r_{10}) + \mu(r_{00} + \beta r_{01})}{s^2(s + \beta + \mu)} \quad (2.2.10)$$

On inverting (2.2.9) and (2.2.10), we get

$$\begin{aligned} V_0(t) = & \frac{1}{\beta + \mu} [\{ (r_{00} + \beta r_{01})\mu + (r_{11} + \mu r_{10})\beta \} t + (r_{00} + \beta r_{01})] \\ & - \frac{1}{(\beta + \mu)^2} [\{ (r_{00} + \beta r_{01})\mu + (r_{11} + \mu r_{10})\beta \} \\ & - \beta(r_{11} + \mu r_{10} - r_{00} - \beta r_{01})e^{-(\beta + \mu)t}] \end{aligned} \quad (2.2.11)$$

$$\begin{aligned} V_1(t) = & \frac{1}{\beta + \mu} [\{ (r_{00} + \beta r_{01})\mu + (r_{11} + \mu r_{10})\beta \} t + (r_{11} + \mu r_{10})] \\ & - \frac{1}{(\beta + \mu)^2} [\{ (r_{00} + \beta r_{01})\mu + (r_{11} + \mu r_{10})\beta \} \\ & - \mu(r_{00} + \beta r_{01} - r_{11} - \mu r_{10})e^{-(\beta + \mu)t}] \end{aligned} \quad (2.2.12)$$

In order to exhibit transient behaviour of the system and thus usage of results arrived above in (2.2.11) and (2.2.12), we give below an illustration.

Illustration

Let us specify various parameters for the purpose of numerical illustration. We give the results in the subsequent tables (2.2.1-2.2.4) for different values of β , μ , r_{00} , r_{11} , r_{01} and r_{10}

Define $V_0^I(t) = V_0(t+1) - V_0(t)$

and $V_1^I(t) = V_1(t+1) - V_1(t)$

From table 2.2.1, it may be observed if the system starts from operable state at $t=0$ it achieves steady-state after 13 units of time and the steady-state expected profit is 3.5238. If the system starts from failed state it achieves steady-state after 14 units of time and the steady-state expected profit is 3.5238.

t	$V_0(t)$	$V_0^I(t)$	$V_1(t)$	$V_1^I(t)$
0	0	13.2600	0	-27.6320
1	13.2600	6.9308	-27.6320	-7.3788
2	20.1908	4.7161	-35.0108	-0.2914
3	24.9069	3.9410	-35.3022	2.1887
4	28.8479	3.6698	-33.1135	3.0567
5	32.5177	3.5749	-30.0568	3.3603
6	36.0926	3.5417	-26.6965	3.4666
7	39.6343	3.5301	-23.2299	3.5038
8	43.1644	3.5260	-19.7261	3.5168
9	46.6904	3.5246	-16.2093	3.5213
10	50.2150	3.5240	-12.6880	3.5230
11	53.7390	3.5239	-9.1650	3.5235
12	57.2629	3.5239	-5.6415	3.5237
13	60.7868	3.5238	-2.1178	3.5237
14	64.3106	3.5238	1.4059	3.5238
15	67.8344	3.5238	4.9297	3.5238
16	71.3582	3.5238	8.4535	3.5238
17	74.8820		11.9773	

$$\beta = 0.25, \mu = 0.8, r_{00} = 20.0, r_{11} = -50.0, r_{01} = -3.0, r_{10} = 4.0$$

Table 2.2.1

It may be observed that initially there is steep fall in $V_0^I(t)$, whereas, as the time progresses, this decrease reduces and stabilizes to a fixed value

which is called steady-state expected profit of the system, Kumar (1976). Further, in the beginning $V_1^I(t)$ increases rapidly and then it stabilizes to the same fixed value viz., expected steady-state profit.

From table 2.2.2, it may be observed that steady-state is achieved after 11 units of time if system starts in operable state and the expected profit is 5.9048. Further, it may be noted that when earning in the failed state increases (cost spent on repairs decreases), expected profit increases and a reduction of 20% in repairs ensures 67.5% increase in the expected profit.

t	$V_0(t)$	$V_0^I(t)$	$V_1(t)$	$V_1^I(t)$
0	0	14.1668	0	-20.5340
1	14.1668	8.7960	-20.5340	-3.3472
2	22.9628	6.9166	-23.8812	2.6672
3	29.8794	6.1982	-21.2140	4.7718
4	36.1382	6.0286	-16.4422	5.5083
5	42.1668	5.9481	-10.9339	5.7660
6	48.1149	5.9200	-5.1679	5.8561
7	54.0349	5.9100	0.6882	5.8878
8	59.9449	5.9067	6.5760	5.8988
9	65.8516	5.9054	12.4748	5.9027
10	71.7570	5.9050	18.3775	5.9040
11	77.6620	5.9048	24.2815	5.9045
12	83.5668	5.9048	30.1860	5.9047
13	89.4716	5.9048	36.0907	5.9047
14	95.3764	5.9047	41.9954	5.9048
15	101.2811	5.9048	47.9002	5.9047
16	107.1859	5.9048	53.8049	5.9048
17	113.0907	5.9047	59.7097	5.9048
18	118.9954		65.6145	

$$\beta = 0.25, \mu = 0.8, r_{00} = 20.0, r_{11} = -40.0, r_{01} = -3.0, r_{10} = 4.0$$

Table 2.2.2

t	$V_0(t)$	$V_0^I(t)$	$V_1(t)$	$V_1^I(t)$
0	0	17.8065	0	-26.1809
1	17.8065	10.9983	-26.1809	-4.3946
2	28.8048	8.6159	-30.5755	3.2293
3	37.4207	7.7821	-27.3462	5.8971
4	45.2028	7.4904	-21.4491	6.8308
5	52.6932	7.3883	-14.6183	7.1575
6	60.0815	7.3525	-7.4608	7.2718
7	67.4340	7.3401	-0.1890	7.3117
8	74.7741	7.3357	7.1227	7.3258
9	82.1098	7.3342	14.4485	7.3307
10	89.4440	7.3336	21.7792	7.3324
11	96.7776	7.3334	29.1116	7.3330
12	104.1110	7.3334	36.4446	7.3332
13	111.4444	7.3333	43.7778	7.3333
14	118.7777	7.3334	51.1111	7.3333
15	126.1111	7.3333	58.4444	7.3333
16	133.4444	7.3333	65.7777	7.3334
17	140.7777		73.1111	

$$\beta = 0.25, \mu = 0.8, r_{00} = 25.0, r_{11} = -50.0, r_{01} = -3.0, r_{10} = 4.0$$

Table 2.2.3

In this case, steady-state is achieved at $t=13$. Further, it may be noted that an increase of 25% in the earning rate of the system while it is operative, there is 108.1% increase in steady-state profit of the system.

t	$V_0(t)$	$V_0^I(t)$	$V_1(t)$	$V_1^I(t)$
0	0	13.6488	0	-23.5955
1	13.6488	8.3342	-23.5955	-2.3366
2	21.9830	6.8114	-25.9321	3.7542
3	28.7944	6.3752	-22.1779	5.4991
4	35.1696	6.2502	-16.6788	5.9995
5	41.4198	6.2144	-10.6793	6.1424
6	47.6342	6.2041	-4.5369	6.1835
7	53.8383	6.2012	1.6466	6.1953
8	60.0395	6.2003	7.8419	6.1986
9	66.2398	6.2001	14.0405	6.1996
10	72.4399	6.2000	20.2401	6.1999
11	78.6399	6.2001	26.4400	6.2000
12	84.8400	6.2000	32.6400	6.2000
13	91.0400	6.2000	38.8400	6.2000
14	97.2400	6.2000	45.0400	6.2000
15	103.4400		51.2400	

$$\beta = 0.25, \mu = 1.0, r_{00} = 20.0, r_{11} = -50.0, r_{01} = -3.0, r_{10} = 4.0$$

Table 2.2.4

Table 2.2.4 shows that faster repair results in increase in profit. An increase in repair rate by 25% gives an increase of about 76% in steady-state profit of the system.

We now depict transient behaviour of the economics of the system so as to bring into light some interesting facts which may be helpful for systems which are operated for a short time span. Behaviour of $V_0(t)$ with respect to variation in t has been given in Fig. 2.2.1. Some of the observations are as under :

- (i) The total earning of the system increases initially. As time progresses, the earning reaches a maximum limit and then starts

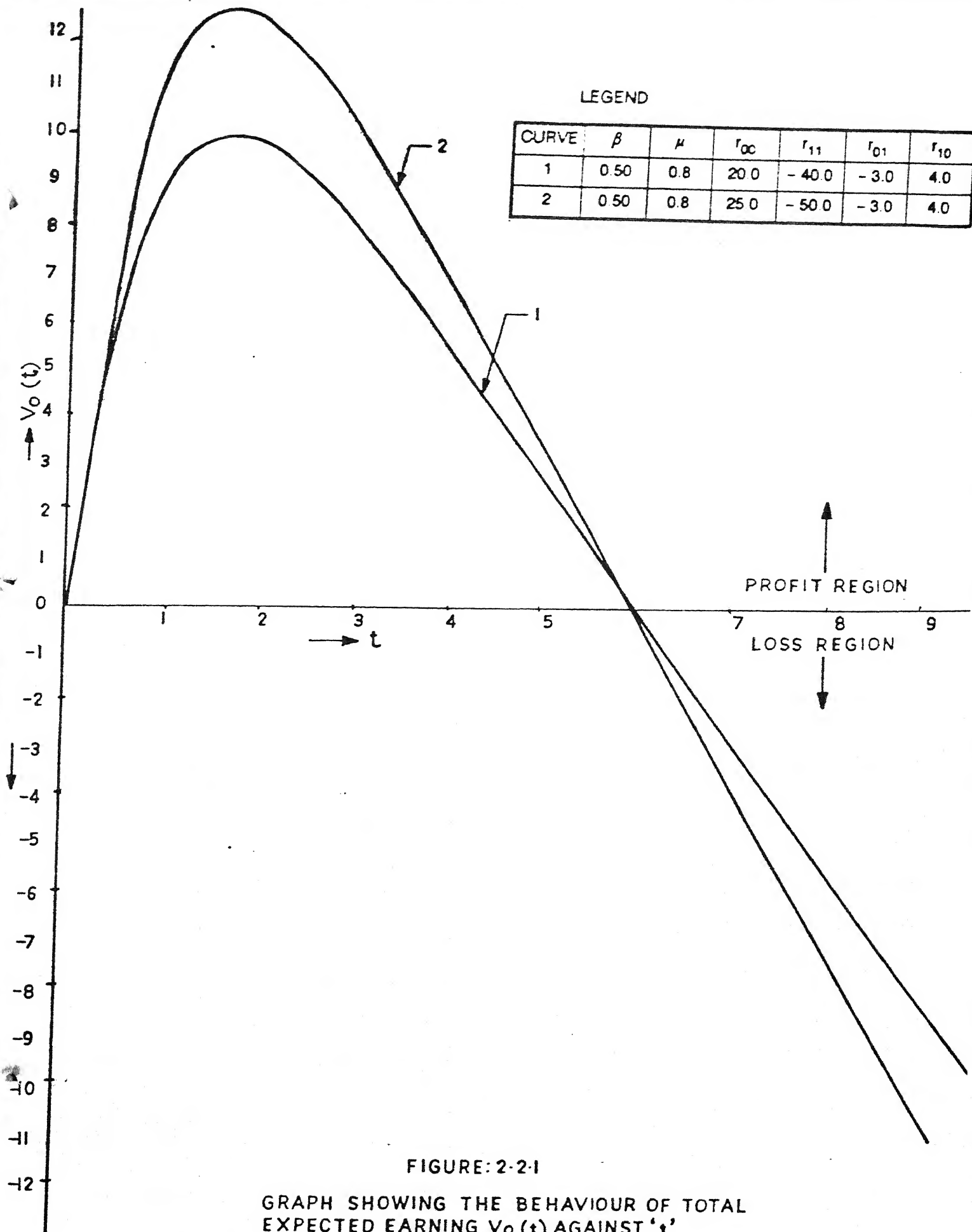


FIGURE: 2-2-1

GRAPH SHOWING THE BEHAVIOUR OF TOTAL EXPECTED EARNING $V_o(t)$ AGAINST 't' (WHEN t IS NOT LARGE ENOUGH)

decreasing. At first sight it may look ridiculous that earning of the system in time t decreases as the time interval t increases. But, a close look in the system structure parameters viz., failure rate, repair rate etc. reveals that the earning does not solely depend upon the time but it is influenced by several other factors including failure and repair rates of the system. When the earning in a larger interval reduces, this merely implies that a major share of the money is spent on repairs of the system.

- (ii) Further, just after 6 units of time, the system starts running in loss and it continues to be in this region in steady-state. More elaborately, after initial adjustments, the system runs in loss. This happens because of the choice of cost parameters and system parameters.

Expected Profit Optimization

The expected profit per unit time denoted by g from (2.2.11) or (2.2.12) after allowing t tend to infinity can be easily obtained. Thus, we get

$$g = \frac{(r_{00} + \beta r_{01})\mu + (r_{11} + \mu r_{10})\beta}{\beta + \mu} \quad (2.2.13)$$

The expected profit g , thus depends upon failure rate β , repair rate μ and r_{ij} ($i, j=0, 1$). It may be observed that it will be of interest to obtain the value for the repair rate which maximizes the expected profit when r_{ij} and β are given (fixed). For the purpose, let us consider

$$r_{00} = f_1(\beta), r_{01} = f_2(\beta), r_{10} = f_3(\mu), r_{11} = f_4(\mu) \quad (2.2.14)$$

Further, if we assume

$$\begin{aligned}
 r_{00} &= f_1(\beta) = \frac{c_1}{\beta}, & c_1 \geq 0, \beta > 0 \\
 r_{01} &= f_2(\beta) = A, & A \leq 0 \\
 r_{10} &= f_3(\mu) = B, & B \geq 0 \\
 r_{11} &= f_4(\mu) = \alpha + c_2\mu + c_3\mu^2, & c_2, c_3 \leq 0, \alpha \text{ is a constant}
 \end{aligned} \tag{2.2.15}$$

On making use of (2.2.15) in (2.2.13) the expected profit g takes the form

$$g = \frac{a\mu^2 + b\mu + c}{\mu + d} \tag{2.2.16}$$

where

$$a = \beta c_3, \quad b = \frac{c_1}{\beta} + \beta(A + B + c_2), \quad c = \alpha\beta, \quad d = \beta.$$

The value of μ which maximizes the expected profit in (2.2.16) is obtained by differentiating g in (2.2.16) with respect to μ and equating it to zero i.e.,

$$\begin{aligned}
 \frac{dg}{d\mu} &= \frac{a\mu^2 + 2ad\mu + bd - c}{(\mu + d)^2} = 0 \\
 \Rightarrow a\mu^2 + 2ad\mu + bd - c &= 0
 \end{aligned} \tag{2.2.17}$$

The optimum value of repair rate μ denoted by μ^* which maximizes g denoted by g^* is given by

$$\mu^* = \frac{-2ad \pm \sqrt{(2ad)^2 - 4a(bd - c)}}{2a} = \frac{-ad \pm \sqrt{\omega}}{a} \tag{2.2.18}$$

where

$$\omega = (ad)^2 - a(bd - c)$$

Since μ^* is finite and cannot be negative, being repair rate, to decide μ^* gives maximum we have to further examine the sign of $\left. \frac{d^2g}{d\mu^2} \right|_{\mu = \mu^*}$

- (i) If $\mu^* > 0$ and $\left. \frac{d^2g}{d\mu^2} \right|_{\mu = \mu^*} < 0$, μ^* provides maximum g i.e., the sufficient condition for $\mu^* > 0$ to be absolute maximum point for g is

$$\beta^3 c_3 + \beta^2(A + B + c_2) + \alpha\beta - c_1 < 0.$$

- (ii) However, if $\mu^* > 0$ and $\left. \frac{d^2g}{d\mu^2} \right|_{\mu = \mu^*} > 0$, μ^* is the minimum point for g i.e., when

$$\beta^3 c_3 + \beta^2(A + B + c_2) + \alpha\beta - c_1 > 0$$

and absolute maximum is achieved at $\mu^* = 0$ for which $g^* = \alpha$.

We now proceed to illustrate the use of results in (2.2.18). We present optimum repair rate μ^* against various values of failure rates, β for different sets of constants α , c_1 , c_2 , c_3 , A and B (where $\alpha = 0$) in the Figure 2.2.2. Some of observations are :

- (i) It may be observed that in general μ^* decreases as β increases. Further the decrease in the value of μ^* for small (initial) values of β is quite substantial as compared to the rate of decrease in the value of μ^* for large values of β . To elaborate this point

LEGEND

CURVE	A	B	C ₁	C ₂	C ₃
1	-4	6	10	-10	5
2	-4	6	15	-10	-5
3	-4	6	10	-6	-5
4	-4	6	10	-10	-3
5	-4	10	10	-10	-5
6	-2	6	10	-10	-5

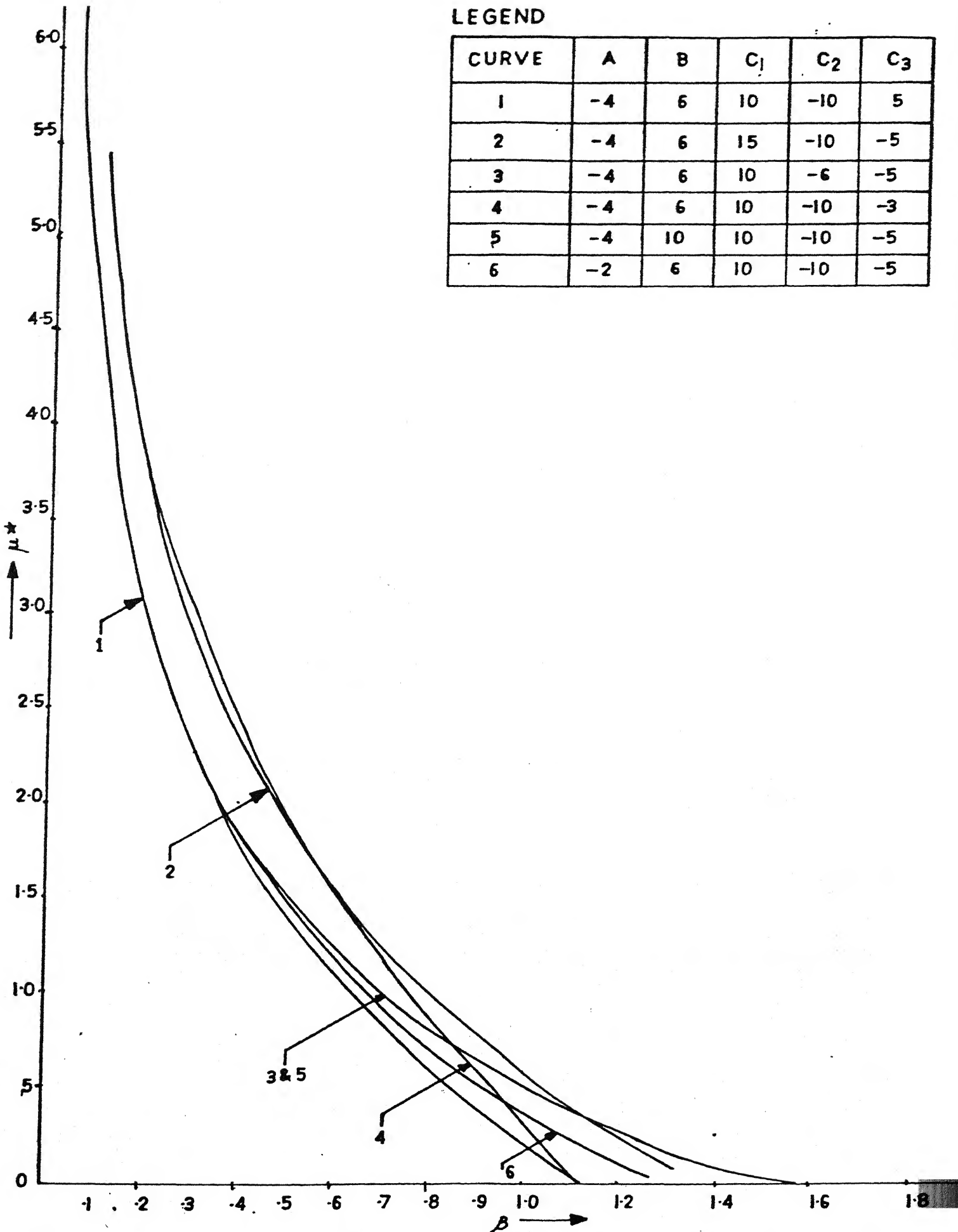


FIGURE: 2-2-2: GRAPH SHOWING THE VARIATION OF OPTIMUM REPAIR RATE (r^*) AGAINST FAILURE RATE (β)

further, we present the results in table 2.2.4a corresponding to curve 1 in Figure 2.2.2. From the same table we observe that the expected profit g^* also behaves in similar manner. Behaviour of other curves also follow the same pattern.

β	μ^*	$\mu^*(\beta) - \mu^*(\beta + .1)$	g^*	$g^*(\beta) - g^*(\beta + .1)$
0.05	6.27	2.80	196.47	136.21
0.15	3.47	0.95	60.26	28.56
0.25	2.52	0.57	31.70	12.74
0.35	1.95	0.42	18.96	7.23
0.45	1.53	0.33	11.73	4.54
0.55	1.20	0.28	7.19	2.97
0.65	0.92	0.25	4.22	1.95
0.75	0.67	0.21	2.27	1.21
0.85	0.46	0.19	1.06	0.70
0.95	0.27	0.17	0.36	0.31
1.05	0.10		0.05	

Table 2.2.4a

- (ii) With the increase in c_1 , the value of μ^* increases but shows a steep increase for smaller values of β , for example

β	μ^* from curve (1)	μ^* from curve (2)	Difference
0.25	2.51	3.16	0.65
1.00	0.18	0.55	0.37

- (iii) With the increase in c_2 , the value of μ^* first increases slowly and then significantly, for example

β	μ^* from curve (1)	μ^* from curve (3)	Difference
0.25	2.51	2.55	0.04
1.00	0.18	0.48	0.30

- (iv) With the increase in c_3 , the value of μ^* first increases significantly

and then slowly, for example

β	μ^* from curve (1)	μ^* from curve (4)	Difference
0.25	2.51	3.30	0.79
1.00	0.18	0.29	0.11

(v) With the increase in A, the value of μ^* first increases slowly and then significantly, for example

β	μ^* from curve (1)	μ^* from curve (6)	Difference
0.25	2.51	2.54	0.03
1.00	0.18	0.34	0.16

Further, if one is confronted with the problem of making a choice regarding installing of a production system whose various parameters like failure rate β , repair rate μ and the earning rates in operable state and failed state, transition costs for making a transition from operable state to failed state and vice versa i.e., r_{ij} are specified. Then one should choose a system which gives more expected profit than others making use of (2.2.13). In what follows we present the table 2.2.5 for given system under consideration.

System								
Parameter	1	2	3	4	5	6	7	8
β	0.5	0.6	0.7	1.0	0.5	0.8	1.0	1.0
μ	1.5	1.3	1.8	1.5	1.5	2.0	2.0	2.5
r_{00}	20.0	25.0	30.0	40.0	35.0	50.0	55.0	55.0
r_{11}	-30.0	-30.0	-35.0	-50.0	-40.0	-65.0	-70.0	-75.0
r_{01}	-4.0	-4.0	-2.0	-2.0	-3.0	-2.0	-3.0	-4.0
r_{10}	3.0	3.0	3.0	3.0	4.0	5.0	4.0	2.0
g	7.12	7.22	12.30	4.60	16.62	18.85	14.00	16.42

Table 2.2.5

Thus we find from the above table that the expected profit g is maximum for the 6th system and therefore, management should opt for this system.

2.3 Model 2 : PROFIT EVALUATION IN A SINGLE UNIT SYSTEM WITH A LESS PRODUCTIVE STATE

In this section we discuss a model of a single unit consisting of two modules. The failure of one module brings the system in a less productive state whereas the failure in the other module brings the system to failed state.

System Model

- (i) There is 1-unit system consisting of type I module and type II module.
- (ii) From operable state, system goes to failed state on the failure of type I module.
- (iii) From operable state, system goes to less productive state on the failure of type II module.
- (iv) From less productive state system goes to failed state on the failure of type I module.
- (v) In the failed state, repairs are carried out to bring the system back to operable state.
- (vi) Failure time distributions of type I and type II modules are assumed to be exponential.
- (vii) Repair time distribution is assumed to be general.
- (viii) In failed state, if there is any failed module of type II, it is repaired free of cost together with repair of type I module.
- (ix) After repair, system acts as new one.

- (x) All probability distributions are assumed to be s-independent.
- (xi) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward (cost) whenever the system changes its state.

Define the following system states to identify the system at any instant.

System states and Transitions

S_0 : Operable or fully productive.

S_1 : Less productive, type II module is in failed state.

S_2 : Failed state, type I module or type I and type II modules are in failed state and the system is under repair.

The system is up in S_0 and S_1 , and it is down in S_2 . The transitions between states are given in Figure 2.3.1.

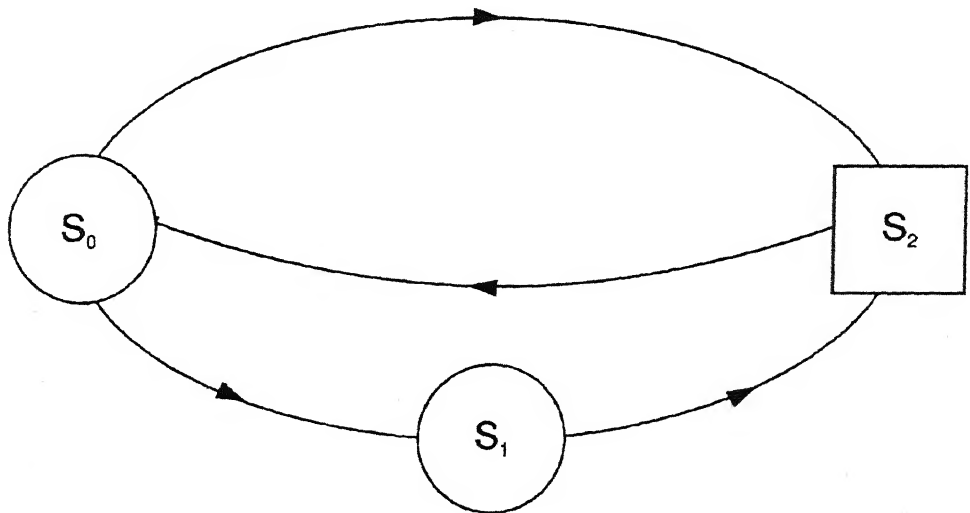


Figure 2.3.1. Transitions between various states

NOTATION

- β_1 : constant failure rate of Type I module.
 β_2 : constant failure rate of Type II module.
 $g(t)$: pdf of repair time
 μ : expected time for repair
 \bar{T}_i : mean unconditional waiting time in S_i ($i = 0, 1, 2$)
 p_{ij} : one step transitional probability from S_i to S_j
 π_i : steady-state probability in S_i ($i = 0, 1, 2$)
 r_{ij} : transition cost for a transition from S_i to S_j (for $i \neq j$).
 r_{ii} : cost per unit time or earning rate in S_i
 g : total expected earning per unit time in steady-state or steady-state expected profit.

Expected Profit of the System

Observing state transitions from figure 2.3.1, various state transition probabilities can be written as :

$$\begin{aligned}
 p_{01} &= \int_0^{\infty} \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} dt = \frac{\beta_2}{\beta_1 + \beta_2} \\
 p_{02} &= \int_0^{\infty} \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{\beta_1}{\beta_1 + \beta_2} \\
 p_{20} &= \int_0^{\infty} g(t) dt = 1 \\
 p_{12} &= \int_0^{\infty} \beta_1 e^{-\beta_1 t} dt = 1
 \end{aligned}
 \tag{2.3.1}$$

Further, it is easy to see that

$$\begin{aligned}\bar{T}_0 &= \int_0^{\infty} e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{1}{\beta_1 + \beta_2} \\ \bar{T}_1 &= \int_0^{\infty} e^{-\beta_1 t} dt = \frac{1}{\beta_1} \\ \bar{T}_2 &= \frac{1}{\mu}\end{aligned}\tag{2.3.2}$$

Steady-state probabilities π_i can be obtained by following relations of Howard (1971) and p_{ij} from (2.3.1)

$$\pi_j = \sum_i p_{ij} \pi_i \tag{2.3.3}$$

and

$$\sum_i \pi_i = 1 \tag{2.3.4}$$

Therefore,

$$\pi_0 = \frac{\beta_1 + \beta_2}{2\beta_1 + 3\beta_2}, \pi_1 = \frac{\beta_2}{2\beta_1 + 3\beta_2}, \pi_2 = \frac{\beta_1 + \beta_2}{2\beta_1 + 3\beta_2} \tag{2.3.5}$$

Following Howard (1971) expected profit g can be written as

$$g = \frac{\sum_i \pi_i T_i q_i}{\sum_i \pi_i T_i} \tag{2.3.6}$$

where

$$q_i = r_{ii} + \frac{1}{T_i} \sum_j p_{ij} r_{ij}$$

Substituting values of p_{ij} from (2.3.1), \bar{T}_i from (2.3.2) and π_i (2.3.5) in (2.3.6) and after simplification g can be obtained as

$$g = \frac{\mu\beta_1(r_{00}+\beta_2r_{01}+\beta_1r_{02})+\mu\beta_2(r_{11}+\beta_1r_{12})+(\beta_1+\beta_2)\beta_1(r_{22}+\mu r_{20})}{\mu(\beta_1+\beta_2)+\beta_1(\beta_1+\beta_2)} \quad (2.3.7)$$

Profit Optimization

Let the repair rate μ of the system is controllable variable, expected profit, g can be optimized for two cases as under :

Case I

When the earning in the failed state is a continuous function of repair rate. Let us specify r_{ij} and r_{ii} as follows :

$$\begin{aligned} r_{00} &= A, & A &> 0, \\ r_{11} &= B, & A &> B > 0 \\ r_{22} &= \alpha + C_1\mu + C_2\mu^2, & \alpha &\leq 0; C_1, C_2 < 0 \end{aligned} \quad (2.3.8)$$

and

$$r_{ij} = 0 \text{ for } i \neq j$$

Substituting these value of r_{00} , r_{11} , r_{22} from (2.3.8) in (2.3.7) and after simplification we get

$$g = \frac{K\mu^2 + L\mu + N}{P\mu + Y} \quad (2.3.9)$$

where

$$\begin{aligned}
K &= (\beta_1 + \beta_2)\beta_1 C_2 \\
L &= \beta_1 A + \beta_2 B + (\beta_1 + \beta_2)\beta_1 C_1 \\
N &= (\beta_1 + \beta_2)\beta_1 \alpha \\
P &= (\beta_1 + \beta_2) \\
Y &= (\beta_1 + \beta_2)\beta_1
\end{aligned} \tag{2.3.10}$$

To determine the value of μ which maximizes g for given β_1, β_2 and known values of cost parameters, we differentiate g in (2.3.9) with respect to μ and equate it to zero i.e., $\frac{dg}{d\mu} = 0$, which gives the value of μ denoted by μ^* maximizing g denoted by g^* , so

$$\mu^* = \frac{-KY \pm \sqrt{K^2 Y^2 - KP(LY - NP)}}{KP} \tag{2.3.11}$$

Since μ^* is finite and cannot be negative, being repair rate, to decide μ^* gives maximum, we have to further examine the sign of $\left. \frac{d^2 g}{d\mu^2} \right|_{\mu = \mu^*}$

(i) If $\mu^* > 0$ and $\left. \frac{d^2 g}{d\mu^2} \right|_{\mu = \mu^*} < 0$, μ^* provides absolute maximum g^* i.e., the sufficient condition for $\mu^* > 0$ to be absolute maximum point for g is $(\beta_1 + \beta_2)(\beta_1^2 C_2 - \beta_1 C_1 + \alpha) < A\beta_1 + B\beta_2$

(ii) However, if $\mu^* > 0$ and $\left. \frac{d^2 g}{d\mu^2} \right|_{\mu = \mu^*} > 0$, μ^* is the minimum point i.e., when $(\beta_1 + \beta_2)(\beta_1^2 C_2 - \beta_1 C_1 + \alpha) > A\beta_1 + B\beta_2$, μ^* is the minimum point and absolute maximum is achieved at $\mu^* = 0$

for which $g^* = \alpha$.

Illustration

In what follows we shall illustrate the results with the help of numerical example. Let us consider a system for which the various values of parameters are given as below :

$$\beta_1 = 0.1, A = 50.0, \alpha = -50.0, C_2 = -2.0$$

$$\beta_2 = 0.2, B = 40.0, C_1 = -4.0$$

On substituting these values $\beta_1, \beta_2, A, B, \alpha, C_1$ and C_2 in (2.3.10) and (2.3.11), we obtain $\mu^* = 6.916$

For $\mu^* = 6.916$, g^* can be obtained from (2.3.9) i.e.,

$$g^* = 40.245.$$

Case II

When the earning in the failed state is a discrete function of repair rate.

In many practical situations the earning of the system in failed state i.e., r_{22} is not a continuous function of the repair rate μ and in fact r_{22} is discrete function of repair rate μ .

Let the management have the option to choose a repair policy out of a number of given repair policies (repair rates) and their corresponding earning rates (r_{22}) in failed states.

In such cases, g is computed for each μ and corresponding r_{22} from

(2.3.7) and that repair policy is chosen which gives more expected profit than others.

In case more than one repair policy gives the same amount of expected profit, any one of them can be selected.

Illustration

In what follows we shall illustrate with a numerical example. Let the values of the various parameters of the system under consideration are specified as below :

$$r_{00} = 50.0, \beta_1 = 0.1, r_{01} = -2.0, r_{12} = -2.0$$

$$r_{11} = 40.0, \beta_2 = 0.2, r_{02} = -3.0, r_{20} = 4.0.$$

Then the values of g computed from (2.3.7) for known values of repair rates, μ and r_{22} are presented in the table 2.3.1.

μ	1.0	2.0	3.0	4.0	4.5	5.0	7.0
r_{22}	-60.0	-68.0	-75.0	-85.0	-88.0	-105.0	-160.0
g	33.969	38.063	39.548	40.235	40.510	40.457	40.502

Table 2.3.1

From the above table we observe that for repair rate, $\mu = 4.5$, g is maximum and hence the management should follow this repair policy.

2.4 Model. 3 : PROFIT CONSIDERATIONS IN A SINGLE UNIT SYSTEM WITH MINOR REPAIR

System Model

- (i) There is a 1-unit system. The system can go to either failed

state or less productive state.

- (ii) When the system goes to less productive state, the management can opt :
 - (a) To carry out minor repair in order to bring the system back to fully productive state.
 - (b) To allow the system to fail.
- (iii) In the failed state repairs are carried out to bring the system back to operable state.
- (iv) Failure-time distributions namely going from operable to failed state and less productive state are assumed to be exponential.
- (v) Repair-time distributions are assumed to be general.
- (vi) All probability distributions are assumed to be s-independent.
- (vii) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward (cost) whenever the system changes its state.

Define the following system states to identify the system at any instant.

System states and Transitions

S_0 : Operable or fully productive

S_1 : Less productive

S_2 : Failed and is under repair

We consider two cases - one under which the system is sent for minor repairs as soon as it enters S_1 and second when the system is allowed to

enter S_2 in due course of time. This is depicted in transitional diagrams (Figure 2.4.1 and 2.4.2).

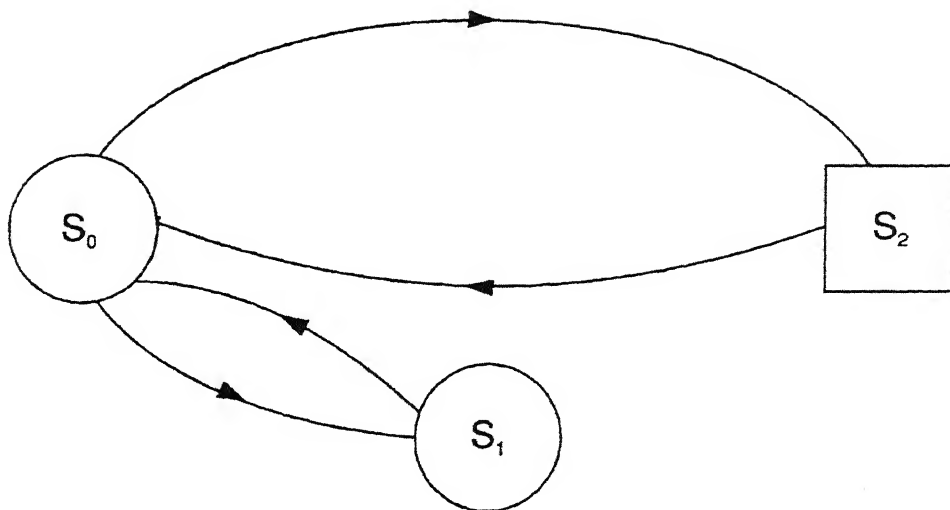


Fig. 2.4.1 Transitions between various states

NOTATION

- β_1 : Constant rate of going from S_0 to S_1
- β_2 : Constant rate of going from S_0 to S_2
- β_3 : Constant rate of going from S_1 to S_2
- $g_1(t)$: pdf of minor repair time
- $g_2(t)$: pdf of repair time
- μ_1 : expected time for minor repair
- μ_2 : expected time for repair
- \bar{T}_i : mean unconditional waiting time in S_i ($i=0, 1, 2$)
- p_{ij} : one step transitional probability from S_i to S_j

- π_i : steady-state probability in S_i ($i=0, 1, 2$)
 r_{ij} : transition reward for a transition from S_i to S_j
 r_{ii} : earning rate in S_i ($i=0, 2$)
 r_{11}^M : earning rate in S_1 when minor repairs are being carried out
 r_{11}^N : earning rate in S_1 when minor repairs are not being carried out
 g_M : expected profit per unit time when the system is under minor repair
 g_N : expected profit per unit time when minor repairs are not carried out in S_1 .

Expected Profit of the System

Case I

Observing state transitions from figure 2.4.1, various transition probabilities can be written as

$$\begin{aligned}
 p_{01} &= \int_0^{\infty} \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{\beta_1}{\beta_1 + \beta_2} \\
 p_{02} &= \int_0^{\infty} \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} dt = \frac{\beta_2}{\beta_1 + \beta_2} \\
 p_{10} &= \int_0^{\infty} g_1(t) dt = 1 \\
 p_{20} &= \int_0^{\infty} g_2(t) dt = 1
 \end{aligned} \tag{2.4.1}$$

Further, it is easy to see that

$$\begin{aligned}
\bar{T}_0 &= \int_0^{\infty} e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{1}{\beta_1 + \beta_2} \\
\bar{T}_1 &= \frac{1}{\mu_1} \\
\bar{T}_2 &= \frac{1}{\mu_2}
\end{aligned}
\tag{2.4.2}$$

Also steady-state probabilities π_i can be obtained by following relations of Howard (1971) and p_{ij} from (2.4.1)

$$\pi_j = \sum_i p_{ij} \pi_i \tag{2.4.3}$$

$$\sum_i \pi_i = 1 \tag{2.4.4}$$

Therefore,

$$\pi_0 = \frac{1}{2}, \quad \pi_1 = \frac{\beta_1}{2(\beta_1 + \beta_2)}, \quad \pi_2 = \frac{\beta_2}{2(\beta_1 + \beta_2)} \tag{2.4.5}$$

Following Howard (1971), expected profit per unit time, g can be written as

$$g = \frac{\sum_i \pi_i \bar{T}_i q_i}{\sum_i \pi_i \bar{T}_i} \tag{2.4.6}$$

where $q_i = r_{ii} + \frac{1}{\bar{T}_i} \sum_j p_{ij} r_{ij}$

Substituting values of p_{ij} from (2.4.1), \bar{T}_i from (2.4.2) and π_i from (2.4.5) in (2.4.6) and after simplification, g_M can be obtained as

$$g_M = \frac{\mu_1 \mu_2 (r_{00} + \beta_1 r_{01} + \beta_2 r_{02}) + \beta_1 \mu_2 (r_{11}^M + \mu_1 r_{10}) + \beta_2 \mu_1 (r_{22} + \mu_2 r_{20})}{\mu_1 \mu_2 + \beta_1 \mu_2 + \beta_2 \mu_1}$$

This can also be written as

$$g_M = \frac{\mu_1 A + B}{\mu_1 D + E} \quad (2.4.7)$$

where $A = \mu_2 (r_{00} + \beta_1 r_{01} + \beta_2 r_{02}) + \beta_1 \mu_2 r_{10} + \beta_2 r_{22} + \beta_2 \mu_2 r_{20}$

$$B = \beta_1 \mu_2 r_{11}^M$$

$$D = \beta_2 + \mu_2$$

$$E = \beta_1 \mu_2$$

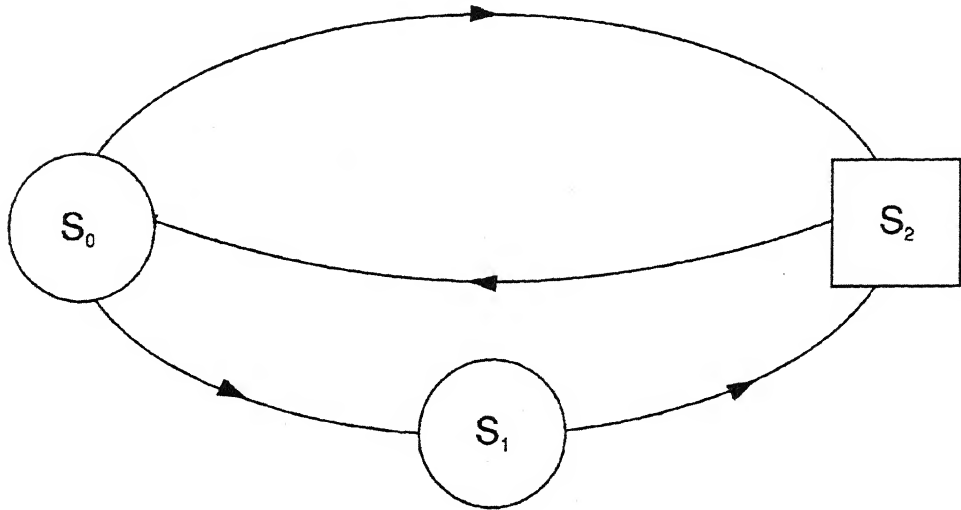


Fig. 2.4.2 Transitions between various states

Case II

Observing state transitions from figure 2.4.2, various transition probabilities can be written as

$$\begin{aligned}
p_{01} &= \int_0^{\infty} \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{\beta_1}{\beta_1 + \beta_2} \\
p_{02} &= \int_0^{\infty} \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} dt = \frac{\beta_2}{\beta_1 + \beta_2} \\
p_{12} &= \int_0^{\infty} \beta_3 e^{-\beta_3 t} dt = 1 \\
p_{20} &= \int_0^{\infty} g_2(t) dt = 1
\end{aligned} \tag{2.4.8}$$

Further, it is easy to see that

$$\begin{aligned}
\bar{T}_0 &= \int_0^{\infty} e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{1}{\beta_1 + \beta_2} \\
\bar{T}_1 &= \int_0^{\infty} e^{-\beta_3 t} dt = \frac{1}{\beta_3} \\
\bar{T}_2 &= \frac{1}{\mu_2}
\end{aligned} \tag{2.4.9}$$

Also steady-state probabilities π_i can be obtained by making use of (2.4.3), (2.4.4) and p_{ij} from (2.4.8).

$$\pi_0 = \frac{\beta_1 + \beta_2}{3\beta_1 + 2\beta_2}, \quad \pi_1 = \frac{\beta_1}{3\beta_1 + 2\beta_2}, \quad \pi_2 = \frac{\beta_1 + \beta_2}{3\beta_1 + 2\beta_2} \tag{2.4.10}$$

Substituting these values of p_{ij} from (2.4.8), \bar{T}_i from (2.4.9) and π_i from (2.4.10) in (2.4.6) and after simplification we get,

$$g_N = \frac{\mu_2 \beta_3 (r_{00} + \beta_1 r_{01} + \beta_2 r_{02}) + \beta_1 \mu_2 (r_{11}^N + \beta_3 r_{12}) + \beta_3 (\beta_1 + \beta_2) (r_{22} + \mu_2 r_{20})}{\mu_2 \beta_3 + \beta_1 \mu_2 + \beta_3 (\beta_1 + \beta_2)}$$

This can also be written as

$$g_N = \frac{\beta_3 F + G}{\beta_3 H + E} \tag{2.4.11}$$

where

$$F = \mu_2(r_{00} + \beta_1 r_{01} + \beta_2 r_{02} + \beta_1 r_{12}) + (\beta_1 + \beta_2)(r_{22} + \mu_2 r_{20})$$

$$G = \beta_1 \mu_2 r_{11}^N$$

$$H = \beta_1 + \beta_2 + \mu_2$$

$$E = \beta_1 \mu_2$$

Decision Criterion Regarding Profit

Now it is evident that when $g_M > g_N$, the management of the system should follow procedures to carry out minor repairs for acquiring more profit.

When $g_M = g_N$, it is immaterial whether minor repairs are carried out in less productive state or the system is allowed to fail.

When $g_M < g_N$, it is advisable, not to follow minor repair procedures.

For $g_M \geq g_N$,

Then, from (2.4.7) and (2.4.11)

$$\frac{\mu_1 A + B}{\mu_1 D + E} \geq \frac{\beta_3 F + G}{\beta_3 H + E}$$

$$\Rightarrow \mu_1 \beta_3 (AH - FD) + \mu_1 (EA - GD) + \beta_3 (BH - FE) \geq GE - BE \quad (2.4.12)$$

Let

$$K = \frac{\mu_1}{\beta_3}$$

Therefore, (2.4.12) reduce to

$$K[\beta_3^2 (AH - FD) + \beta_3 (EA - GD)] \geq GE - BE - \beta_3 (BH - FE).$$

Then, we get critical value of K denoted by K^* as

$$K^* = \frac{(GE-BE)-\beta_3(BH-FE)}{\beta_3^2(AH-FD)+\beta_3(EA-GD)} \quad (2.4.13)$$

For this value of K^* , $g_M = g_N$.

Further, notice if $K < K^*$, it is profitable not to carry out minor repairs, since this leads to less profit.

However, when $K > K^*$, it is profitable to carry out minor repairs.

ILLUSTRATION

Let us consider a system for which the various values of parameters are given as below except μ_1 and β_3

$$\begin{aligned} \beta_1 &= 0.6, & r_{00} &= 10.0, & r_{11}^M &= -5.0, & r_{12} &= -3.0 \\ \beta_2 &= 0.4, & r_{01} &= -2.0, & r_{11}^N &= 5.0, & r_{20} &= 5.0 \\ \mu_2 &= 3.0, & r_{02} &= -4.0, & r_{10} &= 3.0, & r_{22} &= -15.0. \end{aligned}$$

Substituting these values in (2.4.13), the critical values of K^* are given in Table 2.4.1 for various values of β_3

β_3	K^*
0.1	22.505
0.2	10.058
0.3	6.211
0.4	4.406
0.5	3.379
1.0	1.502
1.5	0.949
2.0	0.690
2.5	0.541

Table 2.4.1

Now suppose in the given problem $\beta_3 = 1.0$ and $\mu_1 = 1.0$, then, $K = 1.0$ which is less than K^* ($K^* = 1.502$ from the above table) and therefore, it is profitable if minor repairs are not carried out.

Now suppose $\beta_3 = 1.0$ and $\mu_1 = 2.0$ then $K = 2.0$ which is greater than K^* and therefore, it is profitable to carry out minor repair.

2.5 Model 4 : PROFIT EVALUATION IN A 2-UNIT PARALLEL REDUNDANT SYSTEM

System Model

- (i) The system consists of two units, I and II. Both the units operate simultaneously.
- (ii) Failure-time distributions of units I and II are exponential with rates β_1 and β_2 respectively.
- (iii) Whenever one of the two units fails, the operative unit continues to operate whereas the failed unit undergoes repair.
- (iv) On failure of both units, they undergo repair.
- (v) Repair-time distributions of units I and II are exponential with rates μ_1 and μ_2 respectively.
- (vi) All probability distributions are assumed to be statistically independent.
- (vii) After repair, units act as new ones.
- (viii) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward

(cost) whenever the system changes its state.

Define the following system states to identify the system at any instant.

System States and Transitions

S_0 : Both the units are operating.

S_1 : Unit II is operating while unit I fails and undergoes repair.

S_2 : Unit I is operating while unit II fails and undergoes repair.

S_3 : Both the units are in failed state and are under repair.

The system is up in S_0 , S_1 and S_2 and it is down in S_3 . Transitions between various states are given in Figure 2.5.1.

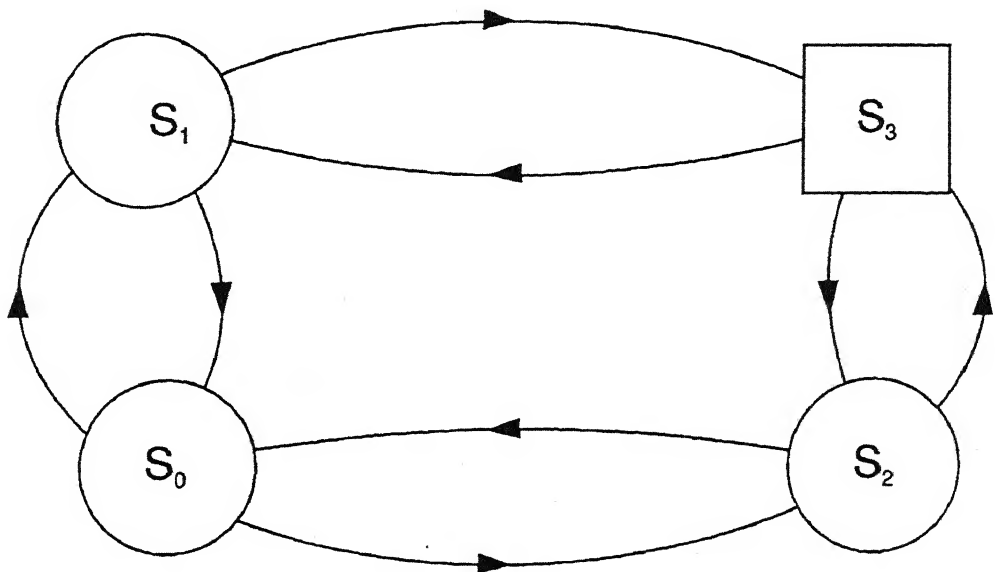


Figure 2.5.1 : Transitions between various states

NOTATION

- β_1 : constant failure rate of Unit I.
 β_2 : constant failure rate of Unit II.
 μ_1 : constant repair rate of Unit I.
 μ_2 : constant repair rate of Unit II.
 r_{ii} : cost per unit time or earning rate in state S_i . ($i=0, 1, 2, 3$).
 r_{ij} : transition cost for a transition from state S_i to S_j , $i \neq j$
 ($i, j = 0, 1, 2, 3$).
 $V_i(t)$: total expected earning of the system for a future period of
 length 't' given that at $t=0$, the system was in S_i
 ($i=0, 1, 2, 3$).

Cost Equations and System Profit

Following Howard (1960), we can write the following cost equations :

$$\begin{aligned}
 V_0(t+\Delta) = & [r_{00}\Delta + V_0(t)][1-\beta_1\Delta][1-\beta_2\Delta] + [r_{01} + V_1(t)] \\
 & \times [1-\beta_2\Delta]\beta_1\Delta + [r_{02} + V_2(t)][1-\beta_1\Delta]\beta_2\Delta
 \end{aligned} \quad (2.5.1)$$

$$\begin{aligned}
 V_1(t+\Delta) = & [r_{11}\Delta + V_1(t)][1-\mu_1\Delta][1-\beta_2\Delta] + [r_{10} + V_0(t)] \\
 & \times [1-\beta_2\Delta]\mu_1\Delta + [r_{13} + V_3(t)][1-\mu_1\Delta]\beta_2\Delta
 \end{aligned} \quad (2.5.2)$$

$$\begin{aligned}
 V_2(t+\Delta) = & [r_{22}\Delta + V_2(t)][1-\mu_2\Delta][1-\beta_1\Delta] + [r_{20} + V_0(t)] \\
 & \times [1-\beta_1\Delta]\mu_2\Delta + [r_{23} + V_3(t)][1-\mu_2\Delta]\beta_1\Delta
 \end{aligned} \quad (2.5.3)$$

$$\begin{aligned}
 V_3(t+\Delta) = & [r_{33}\Delta + V_3(t)][1-\mu_1\Delta][1-\mu_2\Delta] + [r_{31} + V_1(t)] \\
 & \times [1-\mu_1\Delta]\mu_2\Delta + [r_{32} + V_2(t)][1-\mu_2\Delta]\mu_1\Delta
 \end{aligned} \quad (2.5.4)$$

where Δ is a small interval.

$$\text{Initially } V_i(0) = 0 \text{ for } i=0, 1, 2, 3. \quad (2.5.5)$$

Neglecting terms of order of 2 and higher of Δ , and letting $\Delta \rightarrow 0$, we obtain following system of differential equations from (2.5.1) to (2.5.4).

$$\left[\frac{d}{dt} + (\beta_1 + \beta_2)\right]V_0(t) = r_{00} + \beta_1 r_{01} + \beta_2 r_{02} + \beta_1 V_1(t) + \beta_2 V_2(t) \quad (2.5.6)$$

$$\left[\frac{d}{dt} + (\mu_1 + \beta_2)\right]V_1(t) = r_{11} + \mu_1 r_{10} + \beta_2 r_{13} + \mu_1 V_0(t) + \beta_2 V_3(t) \quad (2.5.7)$$

$$\left[\frac{d}{dt} + (\beta_1 + \mu_2)\right]V_2(t) = r_{22} + \mu_2 r_{20} + \beta_1 r_{23} + \mu_2 V_0(t) + \beta_1 V_3(t) \quad (2.5.8)$$

$$\left[\frac{d}{dt} + (\mu_1 + \mu_2)\right]V_3(t) = r_{33} + \mu_2 r_{31} + \mu_1 r_{32} + \mu_2 V_1(t) + \mu_1 V_2(t) \quad (2.5.9)$$

Let $\bar{f}(s)$ denote the Laplace transform of $f(t)$, e.g.,

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \text{Re}(s) > 0.$$

Taking Laplace transform of (2.5.6) to (2.5.9) and employing initial conditions (2.5.5), we get after simplification :

$$\bar{V}_0(s) = \frac{r_{00} + \beta_1 r_{01} + \beta_2 r_{02}}{s(s + \beta_1 + \beta_2)} + \frac{\beta_1 \bar{V}_1(s) + \beta_2 \bar{V}_2(s)}{(s + \beta_1 + \beta_2)} \quad (2.5.10)$$

$$\bar{V}_1(s) = \frac{r_{11} + \mu_1 r_{10} + \beta_2 r_{13}}{s(s + \mu_1 + \beta_2)} + \frac{\mu_1 \bar{V}_0(s) + \beta_2 \bar{V}_3(s)}{(s + \mu_1 + \beta_2)} \quad (2.5.11)$$

$$\bar{V}_2(s) = \frac{r_{22} + \mu_2 r_{20} + \beta_1 r_{23}}{s(s + \beta_1 + \mu_2)} + \frac{\mu_2 \bar{V}_0(s) + \beta_1 \bar{V}_3(s)}{(s + \beta_1 + \mu_2)} \quad (2.5.12)$$

$$\bar{V}_3(s) = \frac{r_{33} + \mu_2 r_{31} + \mu_1 r_{32}}{s(s + \mu_1 + \mu_2)} + \frac{\mu_2 \bar{V}_1(s) + \mu_1 \bar{V}_2(s)}{(s + \mu_1 + \mu_2)} \quad (2.5.13)$$

Further simplification of (2.5.10) to (2.5.13) gives

$$\bar{V}_0(s) = \frac{R_0 A_1 + \beta_1 R_1 A_2 + \beta_2 R_2 A_3 + \beta_1 \beta_2 R_3 A_4}{s^2(s + \beta_1 + \mu_1)(s + \beta_2 + \mu_2)(s + \beta_1 + \mu_1 + \beta_2 + \mu_2)} \quad (2.5.14)$$

where

$$R_0 = r_{00} + \beta_1 r_{01} + \beta_2 r_{02}$$

$$R_1 = r_{11} + \mu_1 r_{10} + \beta_2 r_{13}$$

$$R_2 = r_{22} + \mu_2 r_{20} + \beta_1 r_{23}$$

$$R_3 = r_{33} + \mu_2 r_{31} + \mu_1 r_{32}$$

and

$$A_1 = (s + \mu_1 + \beta_2)(s + \beta_1 + \mu_2)(s + \mu_1 + \mu_2) - \beta_1 \mu_1(s + \mu_1 + \beta_2) - \beta_2 \mu_2(s + \beta_1 + \mu_2)$$

$$A_2 = (s + \beta_1 + \mu_2)(s + \mu_1 + \mu_2) - \beta_1 \mu_1 + \beta_2 \mu_2$$

$$A_3 = (s + \beta_2 + \mu_1)(s + \mu_1 + \mu_2) + \beta_1 \mu_1 - \beta_2 \mu_2$$

$$A_4 = 2s + \beta_1 + \mu_1 + \beta_2 + \mu_2.$$

On inverting (2.5.14) we get

$$\begin{aligned} V_0(t) = & R_0 \left[\frac{a_1 e^{-a_1 t}}{a_2 a_5} + \frac{a_2 e^{-a_2 t}}{-a_1 a_5} - \frac{a_3 e^{-a_3 t}}{a_4} \right] + B_1 \left[\frac{e^{-a_1 t}}{-a_2 a_5} + \frac{e^{-a_2 t}}{a_1 a_5} + \frac{e^{-a_3 t}}{a_4} \right] \\ & + B_2 \left[\frac{1 - e^{-a_3 t}}{a_1 a_2 a_3} + \frac{e^{-a_1 t}}{a_1 a_2 a_5} + \frac{e^{-a_2 t}}{-a_1 a_2 a_5} \right] \\ & + B_3 \left[\frac{a_4 t - a_3}{a_4^2} + \frac{a_2^{-2} e^{-a_2 t} - a_1^{-2} e^{-a_1 t}}{a_5} \right] \end{aligned} \quad (2.5.15)$$

where

$$B_1 = (a_3 + \mu_1 + \mu_2)R_0 + \beta_1 R_1 + \beta_2 R_2$$

$$B_2 = [a_3(\mu_1 + \mu_2) + \beta_1 \beta_2]R_0 + (a_1 + \mu_2)\beta_1 R_1 + (a_2 + \mu_1)\beta_2 R_2 + \beta_1 \beta_2 R_3$$

$$B_3 = R_0 \mu_1 \mu_2 + R_1 \beta_1 \mu_2 + R_2 \mu_1 \beta_2 + R_3 \beta_1 \beta_2$$

and

$$a_1 = \beta_1 + \mu_1, \quad a_2 = \beta_2 + \mu_2, \quad a_3 = a_1 + a_2, \quad a_4 = a_1 a_2, \quad a_5 = a_1 - a_2.$$

Similarly expressions for $V_1(t)$, $V_2(t)$ and $V_3(t)$ can be obtained. The total expected earning of the system per unit time in steady-state or the expected steady-state profit denoted by g may be obtained after allowing t tend to infinity from (2.5.15).

Therefore

$$g = \frac{R_0 \mu_1 \mu_2 + R_1 \beta_1 \mu_2 + R_2 \mu_1 \beta_2 + R_3 \beta_1 \beta_2}{(\beta_1 + \mu_1)(\beta_2 + \mu_2)} \quad (2.5.16)$$

In order to examine economic behaviour of the system in a finite interval of time t (t is not large enough) let us specify various parameters as below :

$$\beta_1 = 0.050, \quad \beta_2 = 0.025, \quad \mu_1 = 1.500, \quad \mu_2 = 1.000,$$

$$r_{00} = 175.0, \quad r_{11} = 50.0, \quad r_{22} = 75.0, \quad r_{33} = -215.0,$$

$$r_{ij} = 0 \text{ for } i \neq j.$$

Substituting these values in (2.5.15) we obtain $V_0(t)$ for various values of t as given in table 2.5.1 :

t	$V_0(t)$	t	$V_0(t)$
0.0	0.0	8.0	1352.33147
0.5	86.61606	9.0	1520.73082
1.0	172.07458	10.0	1689.12988
2.0	341.49547	11.0	1857.52884
3.0	510.19627	12.0	2025.92776
4.0	678.68960	13.0	2194.32666
5.0	847.11942	14.0	2362.72556
6.0	1015.52878	15.0	2531.12446
7.0	1183.93130		

Table 2.5.1

From above table we see that for $t \geq 12$ the increase in value of $V_0(t)$ for every unit increase in value of t is 168.39890 which is nothing but the steady-state expected profit of the system.

Profit Optimization :- Further if one is posed with the problem of selecting failure rate β and repair rate μ from a given set of these parameters along with cost parameters, we should proceed as follows.

- (a) Specify cost matrix (r_{ij}) , values of β and μ corresponding to each alternative.
- (b) Develop cost equations for the system model under consideration and obtain expression for g .
- (c) Choose that alternative taking values of β and μ as decision variables for which g is maximum.

In what follows, we present the results in table 2.5.2 for the system under consideration.

Para meter	Pair No.							
	1	2	3	4	5	6	7	8
β_1	1.0	0.9	0.8	0.7	0.5	0.9	0.8	0.7
β_2	0.9	1.1	0.5	0.5	0.7	1.1	0.5	0.5
μ_1	1.0	1.0	1.0	1.0	1.0	1.8	1.3	1.0
μ_2	1.0	1.0	1.0	1.0	1.0	1.4	1.6	1.3
r_{00}	90.0	80.0	70.0	60.0	50.0	80.0	70.0	60.0
r_{11}	40.0	30.0	30.0	25.0	20.0	30.0	30.0	25.0
r_{22}	39.0	35.0	25.0	20.0	22.0	35.0	25.0	20.0
r_{33}	-99.0	-90.0	-80.0	-70.0	-56.0	-99.0	-80.0	-70.0
r_{01}	-2.0	-2.0	-3.0	-2.0	-1.0	-2.0	-3.0	-2.0
r_{02}	-4.0	-1.0	-2.0	-1.0	-2.0	-1.0	-2.0	-1.0
r_{10}	3.0	2.0	3.0	2.0	2.0	2.0	3.0	2.0
r_{20}	4.0	4.0	2.0	3.0	2.0	4.0	2.0	3.0
r_{13}	-4.0	-4.0	-3.0	-2.0	-2.0	-4.0	-3.0	-2.0
r_{23}	-4.0	-3.0	-2.0	-3.0	-2.0	-3.0	-2.0	-3.0
r_{31}	4.0	4.0	3.0	2.0	3.0	4.0	3.0	2.0
r_{32}	4.0	5.0	2.0	3.0	3.0	5.0	2.0	3.0
g	18.816	16.480	20.643	19.575	20.613	32.591	28.675	21.093

Table 2.5.2

From the above table we observe that for the sixth pair of units steady-state expected profit is maximum.

Particular Case

We now discuss a particular case of the model when both the units are identical. Let the failure and repair rates be β and μ respectively. Define the following three states

State	Condition of the system
S_0	Both the units are operating.
S_1	One unit operates, the other is in failed state and is under repair.
S_2	Both the units are in failed state and are under repair.

The corresponding cost matrix for such a system can be given by

$$R = (r_{ij}) = \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix}$$

Obviously $r_{02} = r_{20} = 0$. Since these transitions are not possible, results for this particular case can easily be obtained for the model discussed above after making the following observation.

Original model	Particular case
β_1, β_2	β
μ_1, μ_2	μ
r_{01}, r_{02}	r_{01}
r_{10}, r_{20}	r_{10}
r_{31}, r_{32}	r_{21}
r_{13}, r_{23}	r_{12}
r_{11}, r_{22}	r_{11}
r_{33}	r_{22}

Therefore

$$g = \frac{1}{(\beta + \mu)^2} [\mu^2(r_{00} + 2\beta r_{01}) + 2\beta\mu(r_{11} + \mu r_{10} + \beta r_{12}) + \beta^2(r_{22} + 2\mu r_{21})] \quad (2.5.17)$$

To see the analytic behaviour of g in (2.5.17), let us consider the following cost structure :

$$r_{00} = A, \quad A > 0$$

$$r_{11} = f_1(\mu) = \alpha_1 + c_1\mu, \quad \alpha_1 > 0, c_1 < 0$$

$$r_{22} = f_2(\mu) = \alpha_2 + c_2\mu, \quad \alpha_2 \geq 0, c_2 < 0$$

and

$$r_{01} = r_{10} = r_{12} = r_{21} = 0.$$

So

$$g = \frac{1}{(\beta + \mu)^2} [\mu^2(A + 2\beta c_1) + \mu(2\beta\alpha_1 + \beta^2 c_2) + \beta^2 \alpha_2] \quad (2.5.18)$$

To determine that value of μ which maximizes expected profit, g in (2.5.18) for a given failure rate and known values of cost parameters, we differentiate g with respect to μ and put equal to zero, i.e., $dg/d\mu = 0$, which gives the value of μ denoted by μ^* which maximizes expected profit, g denoted by g^*

$$\mu^* = \frac{2\beta\alpha_2 - \beta(2\alpha_1 + \beta c_2)}{2(A + 2\beta c_1) - (2\alpha_1 + \beta c_2)}$$

since μ^* is finite and cannot be negative, being repair rate, to decide μ^* gives maximum we have to further examine the sign of $\left. \frac{d^2g}{d\mu^2} \right|_{\mu = \mu^*}$

- (i) If $\mu^* > 0$ and $\left. \frac{d^2g}{d\mu^2} \right|_{\mu = \mu^*} < 0$, μ^* is a maximum point for g i.e., the sufficient condition for $\mu^* > 0$ to be absolute maximum point for g is

$$(A + 2\beta c_1) - (2\alpha_1 + \beta c_2) + \alpha_2 < 0.$$

- (ii) However, if $\mu^* > 0$ and $\left. \frac{d^2g}{d\mu^2} \right|_{\mu = \mu^*} > 0$, μ^* is the minimum point for g i.e., when $(A + 2\beta c_1) - (2\alpha_1 + \beta c_2) + \alpha_2 > 0$

and absolute maximum is achieved at $\mu^* = 0$ for which $g^* = \alpha_2$.

CHAPTER III

OPTIMAL FAILURE ANALYSIS IN

SERIES SYSTEMS

3.1 INTRODUCTION

The maintenance problems related to complex systems with many component modules are quite difficult. Analysis of failures plays an important role in the evaluation studies of maintained systems. Keeping in view the complexity and economics of modern systems, it is of paramount interest to spend on different phases (parts) of a system in an optimal manner. In industrial/defence systems, whenever a failure occurs, it is normal to first decide what type of failure it is. More elaborately, specialization and diversity in repair/or nature of failure demand certain type of activities to be performed during the time lag, when a failure occurs and the instant when it really enters repair. These activities may be termed as 'failure analysis'. A close look at their behaviour leads to the following main activities to be covered under failure analysis.

- (i) **Detection of failure :-** This may be the case with systems which malfunction or operate with reduced efficiency on the occurrence of a failure and the problem is to detect a failure/fault.

The concept of fault detection in maintained systems is important and has been considered by workers in different contexts. In fact, faults may be of 3 types :

- (a) **Identifiable faults :-** are those faults which are identified as and when they occur. In other words, sometimes the subsystems responsible for malfunctioning are observable and failure diagnosis in such cases is straight forward.

- (b) **Unidentifiable faults :-** are those faults which cannot be identified

without any outside intervention. These are hidden faults and may cause serious damages in the long run. Such faults can only be identified/detected by the help of detectors. However, detectors may be used in the case of identifiable faults as well. In other words, there are systems which shut down completely as soon as failure occurs in one of its constituent component modules and such systems do not show any discernible symptoms for detecting the faulty component module. A serially connected heater serves a good example.

(c) **Recoverable faults** :- are those faults which are automatically recovered, once they occur. This feature of maintained systems is taken care of at design stage itself.

(ii) **Identification and location of failure** :- Once the failure is detected the next problem which arises is that of failure identification and its location i.e., to identify the phase or module which has failed.

(iii) **Classification of failure** :- Having arrived at the defective component, one may be interested to classify the failure in one of the several prespecified classes. These classes may depend upon the severity of failure, time/cost to be spent on repair.

(iv) **Administrative time on failure rectification** :- There may be some administrative problems related to failure/repair. The time and money spent on these activities are also a part of failure analysis.

(v) **Transportation of the faulty part for repair** :- The resources

spent on transporting the faulty module (part) to repair shop also constitute a part of failure analysis.

In order to have a good understanding of the genuine requirements of the above activities, let us consider the example of a dockyard in which ships arrive from time to time for refit. The normal procedure adopted is when a ship reports the dockyard for repair, a reviewing committee meets and decides about the type of repairs, time/cost estimates needed on the ship at various work centres in the dockyard. Later on, the ship goes to different work centres and repair is carried out. Here, the time spent on administrative affairs and other activities pertaining to failure analysis is quite substantial and need be included.

The above activities and several other not listed above, need attention of a system analyst responsible for system analysis/evaluation studies of the system under consideration.

A number of authors have concentrated their attention to reliability parameters after including some of the above activities. Kumar and Aggarwal (1978) have analyzed a 2-unit standby system with negligible time spent on fault analysis along with two types of repairs. They obtained MTSF, steady-state availability, absorption probabilities, expected number of visits to a state, second moment of time spent in an up state etc. Kapoor and Kapur (1975b) obtained pointwise availability for two models of a complex system with two classes of components L_1 and L_2 . They assumed a probability distribution for the time spent on post repair. Lal et al. (1980) analyzed a 2-unit standby system with 'analysis time'. This 'analysis time' is constituted either of failure

analysis time or of post repair time. The two models have been analyzed in detail and several parameters of interest are obtained.

There are not many articles on failure-analysis from the view point of cost analysis. Nevertheless, there are articles dealing with optimization problems after taking due consideration of underlying costs.

Gross (1970) considered a 2-unit series systems with failure detection mechanism. He associated a probability of correct detection of faulty component which depends upon the cost spent on the detection mechanism. It is assumed that in case of wrong detection of failure (i.e., assigning failure to a unit which has not failed in reality) there is some extra cost spent on the repair of a good unit. He minimized the expected misclassification cost and obtained a condition under which a detection mechanism would be economically feasible. Later, Kumar (1975) extended the results to an n -unit series system. It is interesting to note that probabilities of detection of failure have been taken as functions of cost. The properties of these cost functions have already been cited in Chapter 1.

Takami et al. (1980) have discussed the problem of allocation of fault detectors to an n -unit series system. The detectors are subject to failure viz., they produce unwanted alarms and they do not produce alarms when they are needed. System equations have been written and steady-state availability has been obtained. Further, assuming cost of detectors and loss due to system failure, optimum number of detectors has been determined to minimize expected cost by applying 0 - 1 programming. Kumar and Kapoor (1979) analyzed a 2-unit series model with detectors which are subject to 2 types of

failures.

All the papers referred above deal with fault detection. As regards fault classification, there has not been much work. Kumar and Kapoor (1981b) have developed a classification rule based on life-time distribution for a 2-unit series system which minimizes expected cost of misclassification. Butterworth (1972) has considered fault testing modules for complex systems assuming a priori probabilities of modules. Park (1982) has developed a mathematical model for diagnosing a faulty component module in an n -unit series system. He has suggested an optimal testing policy based on cost incurred on testing and causative failure probabilities of the underlying modules.

In the present chapter we concentrate on the following important problems pertaining to failure analysis in maintained systems.

1. We have developed a decision rule based on life time distribution of the system that minimizes the expected misclassification cost in an n -unit series system.
2. A 2-unit series system with 2 detectors has been examined with a view to allocate cost to detectors which maximizes the expected steady-state profit of the system.

We discuss the above in detail in the subsequent sections :

3.2 OPTIMAL FAILURE CLASSIFICATION IN AN n -UNIT SERIES SYSTEM

We discuss below the problem of classification of failures in a series system.

Failure Classification :- System Model and Problem Definition

1. There is a series system consisting of n units. Let the units be denoted by $1, 2, \dots, n$. The units are independent and have general distributions for their failure times. Obviously, the system fails upon failure of any one of the n -units.
2. At the time of system failure, the problem is to classify this failure in one of the n classes (mutually exclusive), each class corresponds to a unit in the system. Thus, when a failure is classified into i^{th} class it implies that the system failure has occurred due to failure of the i^{th} unit. While dealing with failure classification, one is prone to the error of classifying a failure. More elaborately, a failure ought to be classified in class i , is classified in class j . This leads to loss, may be termed as 'misclassification cost'.
3. Further, a priori probabilities of failure of units are assumed to be known.

Notation

- i, j : index, which denote units label = $1, 2, \dots, n$.
- C_{ij} : cost of misclassification of a failure, assigning the failure to unit j whereas unit i has failed.
- $f_i(t)$: life-time pdf of unit i .
- R : sample space
- R_i : subset of R which corresponds to failure of unit i .

$$R = \bigcup_{i=1}^n R_i$$

$p_{ij}(R)$: probability of misclassification of a failure due to unit j whereas unit i has failed.

$$= \int_{R_j} f_i(t) dt$$

π_i : a priori probability of failure of unit i .

Optimal Classification : Loss Minimization and Classification Rule Development

The expected loss due to misclassification is given by

$$E(L) = \sum_{i=1}^n \pi_i \left[\sum_{\substack{j=1 \\ j \neq i}}^n C_{ij} p_{ij}(R) \right] \quad (3.2.1)$$

The problem is to choose R_i 's so that $E(L)$ is minimized. Since we know a priori failure probabilities for each unit and corresponding life-time pdfs, we may define the conditional probability of failure due to unit i as

$$\frac{\pi_i f_i(t)}{\sum_{k=1}^n \pi_k f_k(t)} \quad (3.2.2)$$

On classifying the failure due to unit j , the expected loss is

$$\sum_{\substack{i=1 \\ i \neq j}}^n \frac{\pi_i f_i(t)}{\sum_{k=1}^n \pi_k f_k(t)} C_{ij} \quad (3.2.3)$$

This expected loss is to be minimized by the proper choice of j .

Equivalently, the problem is to minimize

$$\sum_{\substack{i=1 \\ i \neq j}}^n \pi_i f_i(t) C_{ij} \quad (3.2.4)$$

The choice of j may be made by evaluating (3.2.4) for all values of j 's and choose the one which gives the minimum. If there are more than one j 's giving the same minimum, any one of them may be chosen. In this manner R_j s are formed. Thus the classification procedure is to classify failure to unit j if it falls in R_j .

In what follows is the following theorem :

Theorem :- If π_i is the a priori probability of a failure due to unit i in an n -unit series system with probability density function $f_i(t)$ and if C_{ij} is the cost of misclassification assigning the failure to unit j whereas unit i has failed then the regions of classification, R_1, R_2, \dots, R_n , that minimizes the expected loss are defined assigning the failure to k belonging to R_k if

$$\sum_{\substack{i=1 \\ i \neq k}}^n \pi_i f_i(t) C_{ik} < \sum_{\substack{i=1 \\ i \neq j}}^n \pi_i f_i(t) C_{ij}, \quad (j = 1, 2, \dots, n, j \neq k) \quad (3.2.5)$$

(If (3.2.5) holds for all j ($j \neq k$) except for h indices and the inequality is replaced by equality for those indices, then this point can be assigned to any one of $h+1$ units).

Discriminatory Power of the Classification Rule :- Now we suggest a method to study the discriminatory power of the classification rule developed above.

Let α_{ij} denote the number of failures known which have occurred due to failure of unit i but which were classified as have occurred due to failure

of unit j . Then the matrix for the classification problem is defined to be $m \times m$ matrix $S = (\alpha_{ij})$. this matrix may be called the Right/Wrong Identification Matrix (RWIM) and is given by

$$S = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2m} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & \alpha_{mm} \end{bmatrix} \quad (3.2.6)$$

Diagonal elements of S denote the number of correct classifications and off diagonal elements denote the number of misclassifications.

Then, we define the normalised RWIM, S_0 as

$$S_0 = (\gamma_{ij})$$

$$\gamma_{ij} = \frac{\alpha_{ij}}{\sum_{j=1}^m \alpha_{ij}}$$

Evidently, the elements of normalised RWIM give proportions of classifications : right and wrong.

For the purpose of testing the discriminatory power of the classification procedure, we make use of a test based on χ^2 distribution. Accordingly, we define

$$Q = \frac{(n-e)^2}{e} + \frac{(\bar{n}-\bar{e})^2}{\bar{e}} \quad (3.2.7)$$

where n and \bar{n} denote the number of right and wrong classifications made by the discriminatory procedure; e and \bar{e} denote the expected number of right

and wrong classifications that would be made if the classifications were made at random. Then, if $N = \sum_{i=1}^m N_i$ denotes the total number of classifications made, and the probability of a successful random classification is $\frac{1}{m}$, then

$$\begin{aligned} n &= \sum_{j=1}^m \alpha_{ij} , & \bar{n} &= N - n \\ e &= \frac{N}{m} , & \bar{e} &= N - \frac{N}{m} \end{aligned} \quad \left| \quad (3.2.8) \right.$$

On substituting (3.2.8) in (3.2.7), we get

$$Q = \frac{(N - nm)^2}{N(m-1)} \quad (3.2.9)$$

a form convenient for numerical evaluation. Now, if we define :

H_0 : classification is made at random

H_1 : classification is made making use of (3.2.5).

Under H_0 , (3.2.9) follows Chi-square distribution with (m^2-1) degrees of freedom.

H_0 will be rejected or accepted accordingly as calculated value of Chi-square, is larger or smaller than the tabulated value.

In order to explain the usage of the results developed above, we now give a numerical example.

Illustration

Let us consider a 3-unit series system with units 1, 2 and 3 and with exponential life-time distributions for the units, i.e.,

$$f_i(t) = \beta_i e^{-\beta_i t}, \quad i = 1, 2, 3$$

Then (3.2.5) reduces to

$$\sum_{\substack{i=1 \\ i \neq k}}^3 \pi_i f_i(t) C_{ik} < \sum_{\substack{i=1 \\ i \neq j}}^3 \pi_i f_i(t) C_{ij}$$

We assign failure to unit 1 if the following set of inequalities are satisfied

$$\pi_2 f_2(t) C_{21} + \pi_3 f_3(t) C_{31} < \pi_1 f_1(t) C_{12} + \pi_3 f_3(t) C_{32}$$

$$\pi_2 f_2(t) C_{21} + \pi_3 f_3(t) C_{31} < \pi_1 f_1(t) C_{13} + \pi_2 f_2(t) C_{23}$$

The failure is assigned to unit 2 if the following set of inequalities are satisfied.

$$\pi_1 f_1(t) C_{12} + \pi_3 f_3(t) C_{32} < \pi_1 f_1(t) C_{13} + \pi_2 f_2(t) C_{23}$$

$$\pi_1 f_1(t) C_{12} + \pi_3 f_3(t) C_{32} < \pi_2 f_2(t) C_{21} + \pi_3 f_3(t) C_{31}$$

The failure is assigned to unit 3 if the following set of inequalities are satisfied.

$$\pi_2 f_2(t) C_{23} + \pi_1 f_1(t) C_{13} < \pi_2 f_2(t) C_{21} + \pi_3 f_3(t) C_{31}$$

$$\pi_2 f_2(t) C_{23} + \pi_1 f_1(t) C_{13} < \pi_1 f_1(t) C_{12} + \pi_3 f_3(t) C_{32}$$

Now for given set of values

$$\beta_1 = 0.185 \quad C_{21} = 20.0, \quad C_{31} = 30.0$$

$$\beta_2 = 0.040 \quad C_{12} = 15.0, \quad C_{32} = 25.0$$

$$\beta_3 = 0.080 \quad C_{13} = 15.0, \quad C_{23} = 20.0$$

We find π_i by

$$\pi_i = \frac{\beta_i}{\sum_{i=1}^3 \beta_i} \quad (i=1, 2, 3) \quad (3.2.10)$$

On substituting values of β_1 , β_2 and β_3 in (3.2.10), we obtain

$$\pi_1 = 0.60656, \quad \pi_2 = 0.13115, \quad \pi_3 = 0.26230$$

Also making appropriate substitutions in inequalities (3.2.5), we see that when $t \geq 40.2360$ component 2 is classified as failed, when $9.3670 \leq t < 40.2360$ component 3 is classified as failed and when $0 \leq t < 9.3670$ component 1 is classified as failed.

Now to test the discriminatory power we say, let the 100 classifications are made which are given in RWIM as below

	1	2	3	Totals
1	10	5	7	22
2	6	25	9	40
3	16	10	12	38

Then, we observe $N=100$, $n=47$, $m=3$

$$Q = \frac{(100 - 3 \times 47)^2}{100(3-1)} = 8.405$$

Also Chi-square at 1% level is 1.646 is certainly significant. Thus the classification procedure certainly does better than chance.

3.3 OPTIMAL ALLOCATION OF COST TO DETECTORS IN A 2-UNIT SERIES SYSTEM

System Model

- (i) There is a 2-unit series system. Let the units be identified as 1 and 2.

- (ii) Each unit is equipped with a separate detector to detect failure. As soon as the system failure occurs i.e., one of 2-units fails, detectors will detect the failures with some prespecified probabilities.
- (iii) There is no interconnection between the detectors. In fact, probability of correct detection is directly related to the cost spent on detectors i.e., more the cost spent on the detector, the better probability of correct detection (smaller probability of wrong detection) will be ensured. Some of the properties of these probabilities are discussed in Kumar (1975).
- (iv) When a unit is wrongly detected as failed, it enters for repair. It is assumed here that the repair is not carried out completely but after a short (random) duration it is identified that there was wrong detection. As such the failed unit goes to repair without any ambiguity now.
- (v) When the system is down, the operative unit cannot fail.
- (vi) It is assumed that detectors do not fail at the time of need.
- (vii) Failure time distributions of units are assumed to be exponential whereas repair time distributions are general.
- (viii) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward (cost) whenever the system changes its state.

Define following system states at any instant.

System states and transitions

- S_0 : Both the units 1 and 2 are in operative state.
- S_1 : Unit 2 fails but unit 1 is detected as failed and goes for repair.
- S_2 : Unit 1 fails but unit 2 is detected as failed and goes for repair.
- S_3 : Unit 2 fails and is detected as failed and goes for repair.
- S_4 : Unit 1 fails and is detected as failed and goes for repair.

The system is up in S_0 and is down in S_i ($i=1, 2, 3, 4$). Transitions between various states are given in figure 3.3.1.

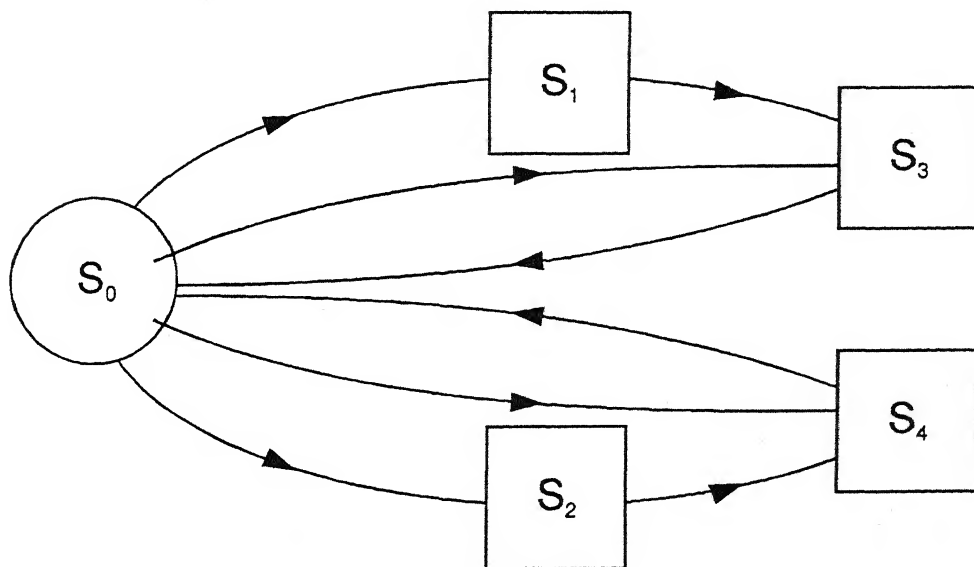


Figure 3.3.1 Transitions between various states

Notation

β_1 : constant failure rate of unit 1

β_2 : constant failure rate of unit 2

- m_1 : mean repair time of unit 1
- m_2 : mean repair time of unit 2
- m_{12} : mean time for which the repair is carried out on unit 1 in S_1 wrongly whereas unit 2 was actually in failed state or mean time taken for detecting unit 2 as failed.
- m_{11} : mean time for which the repair is carried out on unit 2 in S_2 wrongly whereas unit 1 was actually in failed state or mean time taken for detecting unit 1 as failed.
- $p_{12}(x_1)$: probability of detecting unit 2 as failed whereas unit 1 has failed when x_1 units of money are spent on detector attached to unit 1.
- $p_{21}(x_2)$: probability of detecting unit 1 as failed whereas unit 2 has failed when x_2 units of money are spent on detector attached to unit 2.
- $p_{11}(x_1)$: probability of detecting unit 1 as failed when unit 1 fails and x_1 units of money are spent on detector attached to unit 1.
- $p_{22}(x_2)$: probability of detecting unit 2 as failed when unit 2 fails and x_2 units of money are spent on detector attached to unit 2.
- i, j : subscripts that imply system states $i, j = 0, 1, 2, 3, 4$.
- p_{ij} : One step transition probability for making a transition from S_i to S_j
- $P = [p_{ij}]$: transition probability matrix

- I : identity matrix of order 5
 D : $I - P$
 d_i : i^{th} subdeterminant of D , deleting i^{th} row and i^{th} column
 π_i : steady-state probability in $S_i = d_i / \sum_j d_j$
 π : row vector, $(\pi_0, \pi_1, \dots, \pi_4)$
 \bar{T}_i : mean unconditional waiting time in S_i
 r_{ii} : earning rate in S_i
 r_{ij} : transition reward (cost) for making a transition from S_i to S_j
 g : total expected earning per unit time in steady-state or steady-state expected profit.

Expected Profit of the System

Observing state transitions from figure 3.3.1, various state transition probabilities, p_{ij} can be written as

$$\begin{aligned}
 p_{01} &= p_{21}(x_2) \int_0^{\infty} \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} dt = p_{21}(x_2) \frac{\beta_2}{\beta_1 + \beta_2} \\
 p_{02} &= p_{12}(x_1) \int_0^{\infty} \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} dt = p_{12}(x_1) \frac{\beta_1}{\beta_1 + \beta_2} \\
 p_{03} &= p_{22}(x_2) \int_0^{\infty} \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} dt = p_{22}(x_2) \frac{\beta_2}{\beta_1 + \beta_2} \\
 p_{04} &= p_{11}(x_1) \int_0^{\infty} \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} dt = p_{11}(x_1) \frac{\beta_1}{\beta_1 + \beta_2}
 \end{aligned} \tag{3.3.1}$$

Obviously, $p_{13} = p_{24} = p_{30} = p_{40} = 1$

Further, it is easy to see that

$$\begin{aligned} \bar{T}_0 &= \int_0^{\infty} e^{-\beta_2 t} e^{-\beta_1 t} dt = \frac{1}{\beta_1 + \beta_2} \\ \bar{T}_1 &= m_{11}, \quad \bar{T}_2 = m_{12}, \quad \bar{T}_3 = m_2, \quad \bar{T}_4 = m_1 \end{aligned} \quad (3.3.2)$$

Also

$$\begin{aligned} P_{12}(x_1) + P_{11}(x_1) &= 1 \\ P_{21}(x_2) + P_{22}(x_2) &= 1 \end{aligned} \quad (3.3.3)$$

Following Barlow and Proschan (1965),

we know

$$\pi = \pi P \quad (3.3.4)$$

$$\pi(I - P) = 0 \quad (3.3.5)$$

so, we obtain d_i as follows

$$d_0 = 1, \quad d_1 = p_{01}, \quad d_2 = p_{02}, \quad d_3 = p_{01} + p_{03}, \quad d_4 = p_{02} + p_{04} \quad (3.3.6)$$

Also

$$\pi_i = \frac{d_i}{\sum_j d_j} \quad (3.3.7)$$

Following Howard (1971), steady-state expected profit, g can be given by

$$g = \frac{\sum_i \pi_i \bar{T}_i q_i}{\sum_i \pi_i \bar{T}_i} \quad (3.3.8)$$

where

$$q_i = r_{ii} + \frac{1}{\bar{T}_i} \sum_j p_{ij} r_{ij}$$

Substituting (3.3.7) in (3.3.8), we get

$$g = \frac{\sum_i d_i \bar{T}_i q_i}{\sum_i d_i \bar{T}_i} \quad (3.3.9)$$

On making appropriate substitutions in (3.3.9) and taking $r_{ij}=0$ for $i \neq j$, also using (3.3.3) and after simplification g is obtained as

$$g = \frac{r_{00} + \beta_2 r_{33} m_2 + \beta_1 r_{44} m_1 + \beta_2 p_{21}(x_2) r_{11} m_{12} + \beta_1 p_{12}(x_1) r_{22} m_{11}}{1 + \beta_2 [p_{21}(x_2) m_{12} + m_2] + \beta_1 [p_{12}(x_1) m_{11} + m_1]} \quad (3.3.10)$$

If no cost is spent on detectors, it is reasonable to think that on the failure of the system, any unit (1 or 2) can go for repair with equal probability i.e.,

$$p_{12}(0) = p_{21}(0) = p_{11}(0) = p_{22}(0) = \frac{1}{2} \quad (3.3.11)$$

Then

$$g = \frac{r_{00} + \beta_2 r_{33} m_2 + \beta_1 r_{44} m_1 + \frac{\beta_2}{2} r_{11} m_{12} + \frac{\beta_1}{2} r_{22} m_{11}}{1 + \beta_2 [\frac{m_{12}}{2} + m_2] + \beta_1 [\frac{m_{11}}{2} + m_1]} \quad (3.3.12)$$

Optimal Cost Allocation to Detectors

Now, if one is posed with the problem of optimal cost allocation to detectors attached to units 1 and 2 with a view to maximize expected profit, g of the system under given cost constraint, then the problem reduces to

Maximize g in (3.3.10)

subject to

$$x_1 + x_2 = x \quad x_1, x_2, x > 0 \quad (3.3.13)$$

where x_1, x_2 are the costs to be spent on detectors attached with units 1 and 2 respectively, x is the total amount available.

We now discuss the above problem in the light of two cost structures used for failure detection mechanism.

Failure Detection Cost Structure 1

Let

$$p_{12}(x_1) = \frac{1}{2(1+x_1)}$$

and

$$p_{21}(x_2) = \frac{1}{2(1+x_2)}$$

(3.3.14)

It may be noted that (3.3.14) satisfies properties given in Kumar (1975).

On substituting (3.3.14) in (3.3.13), the problem is to maximize g subject to $x_1 + x_2 = x$ where g is given by

$$g = \frac{r_{00} + \beta_2 r_{33} m_2 + \beta_1 r_{44} m_1 + \frac{\beta_2 r_{11} m_{12}}{2(1+x_2)} + \frac{\beta_1 r_{22} m_{11}}{2(1+x_1)}}{1 + \beta_2 m_2 + \beta_1 m_1 + \frac{\beta_2 m_{12}}{2(1+x_2)} + \frac{\beta_1 m_{11}}{2(1+x_1)}} \quad (3.3.15)$$

Observing the equality, $x_1 + x_2 = x$, (3.3.15) reduces to

$$g = \frac{Ax_1^2 + Dx_1 + E}{Bx_1^2 + Fx_1 + G} \quad (3.3.16)$$

where

$$A = -2A_1, \quad D = 2A_1x + D_1 - E_1, \quad E = (1+x)(2A_1 + E_1) + D_1$$

$$B = -2B_1, \quad F = 2B_1x + F_1 - G_1, \quad G = (1+x)(2B_1 + G_1) + F_1$$

and

$$A_1 = r_{00} + \beta_2 r_{33} m_2 + \beta_1 r_{44} m_1, \quad D_1 = \beta_2 r_{11} m_{12}, \quad E_1 = \beta_1 r_{22} m_{11}$$

$$B_1 = 1 + \beta_2 m_2 + \beta_1 m_1, \quad F_1 = \beta_2 m_{12}, \quad G_1 = \beta_1 m_{11}$$

To maximize g in (3.3.16) we differentiate g with respect to x_1 and put $\frac{dg}{dx_1} = 0$, which gives

$$x_1 = \frac{-L \pm \sqrt{L^2 - KM}}{K} \quad (3.3.17)$$

where

$$K = AF - DB, \quad L = AG - EB, \quad M = GD - EF.$$

As x_1 is cost we consider only positive value of x_1 , and denote it by x_1^* .

To examine if x_1^* is maximum point, it is ethical to verify

$$\left. \frac{d^2g}{dx_1^2} \right|_{x_1 = x_1^*} < 0$$

Once x_1^* is determined, x_2^* (the optimal cost spent on the second detector) is obtained as $x_2^* = x - x_1^*$.

Illustration

To illustrate the results numerically, let us specify the various parameters for the problem as under :

$$\begin{aligned}\beta_1 &= 0.2, & \beta_2 &= 0.5, & m_1 &= 0.8, & m_2 &= 0.4, & m_{11} &= 0.1, \\ m_{12} &= 0.2, & r_{00} &= 20.0, & r_{11} &= -22.0, & r_{22} &= -23.0, & r_{33} &= -25.0, \\ r_{44} &= -27.0, & x &= 50.0.\end{aligned}$$

When the above values are substituted in (3.3.17), we get

$$x_1^* = 15.242, \quad x_2^* = 34.758$$

and

$$g = 7.8083$$

On the other hand, if no cost is spent on detectors, (3.3.12) gives

$$g = 6.5845$$

Thus inclusion of detectors ensures an increase of 18.58% in expected profit.

Failure Detection Cost Structure 2

Let

$$p_{12}(x_1) = \frac{1}{2} e^{-a_1 x_1}$$

and

$$p_{21}(x_2) = \frac{1}{2} e^{-a_2 x_2}$$

(3.3.19)

where a_1 and a_2 are constants depending upon the requirement of correct detection assurance.

On substituting (3.3.19) in (3.3.13), the problem is to maximize g subject to $x_1 + x_2 = x$ where g is given by

$$g = \frac{r_{00} + \beta_2 r_{33} m_2 + \beta_1 r_{44} m_1 + \frac{\beta_2}{2} e^{-a_2 x_2} r_{11} m_{12} + \frac{\beta_1}{2} e^{-a_1 x_1} r_{22} m_{11}}{1 + \beta_2 m_2 + \beta_1 m_1 + \frac{\beta_2}{2} e^{-a_2 x_2} m_{12} + \frac{\beta_1}{2} e^{-a_1 x_1} m_{11}} \quad (3.3.20)$$

Observing the equality $x_1 + x_2 = x$, (3.3.20) reduces to

$$g = \frac{A + B e^{-a_1 x_1} + H e^{a_2 x_1}}{E + F e^{-a_1 x_1} + K e^{a_2 x_1}} \quad (3.3.21)$$

where

$$A = r_{00} + \beta_2 r_{33} m_2 + \beta_1 r_{44} m_1, \quad E = 1 + \beta_2 m_2 + \beta_1 m_1, \\ B = \frac{\beta_1 r_{22} m_{11}}{2}, \quad F = \frac{\beta_1 m_{11}}{2}, \quad H = \frac{\beta_2 r_{11} m_{12} e^{-a_2 x}}{2}, \quad K = \frac{\beta_2 m_{12} e^{-a_2 x}}{2}.$$

The value of $a_2 x$ has been bounded for the traceability of results i.e., $|a_2 x| \leq 4$.

To maximize g in (3.3.21) we differentiate g with respect to x_1 and put

$$\frac{dg}{dx_1} = 0, \text{ which gives}$$

$$s_1 e^{-a_1 x_1} + s_2 e^{a_2 x_1} + s_3 e^{(a_2 - a_1) x_1} = 0 \quad (3.3.22)$$

where

$$s_1 = a_1 (AF - EB), \quad s_2 = a_2 (EH - AK), \quad s_3 = (a_1 + a_2) (FH - KB).$$

Let

$$t = e^{a_1 x_1} \text{ and } a_1 = a_2 \text{ in (3.3.22)}$$

Then

$$s_2 t^2 + s_3 t + s_1 = 0 \quad (3.3.23)$$

Solving (3.3.23) for t , we get

$$t = \frac{-s_3 \pm \sqrt{s_3^2 - 4s_1 s_2}}{2s_2}$$

Since x_1 is cost, we consider only that value of t which gives $x_1 > 0$ from

$$x_1 = \frac{\log t}{a_2}$$

where \log is taken to the base e .

Let us denote this value of x_1 by x_1^* .

To decide that x_1^* actually maximizes g in (3.3.21), it is ethical to verify

$$\left. \frac{d^2 g}{dx_1^2} \right|_{x_1 = x_1^*} < 0$$

Once x_1^* is determined, x_2^* (the optimal cost to be spent on second detector) is obtained as $x_2^* = x - x_1^*$.

Illustration

To illustrate the results numerically, let us specify the various parameters as under :

$$\begin{aligned}\beta_1 &= 0.02, & m_1 &= 0.8, & r_{22} &= -230.0, & \beta_2 &= 0.05, & m_2 &= 0.4, \\ r_{33} &= -240.0, & m_{11} &= 0.1, & r_{00} &= 200.0, & r_{44} &= -250.0, & m_{12} &= 0.2, \\ r_{11} &= -220.0, & x &= 1000.0, & a_1 &= a_2 &= .004\end{aligned}$$

When the above substitutions are made to obtain the solution, we get

$$x_1^* = 305.9075, \quad x_2^* = 694.0925$$

and

$$g = 184.3361.$$

On the other hand if no cost is spent on detectors, we find from (3.3.12)

$$g = 182.2168$$

Thus, we observe that employing detectors increases the expected profit of the system.

CHAPTER IV

EVALUATION OF SERIES

SYSTEMS THROUGH DYNAMIC

PROGRAMMING TECHNIQUE

4.1 INTRODUCTION

Designing reliability into a system is often required when the system is to perform an important mission for industry/military or space flight and where malfunction (failure) of a constituent part causes heavy losses in terms of money/time.

In order that a multistage system having many subsystems or stages in series have a failure free operation, the subsystems' reliability should be very high. While dealing with evaluation studies of series systems, one often finds following ways to increase system reliability.

- (a) To increase the reliability of the constituent subsystems (components) i.e., to develop highly reliable components. This in turn increases cost of the system. Further, as the number of series components increases, the overall system reliability decreases. In fact, after a certain stage, even a marginal increase in a component's reliability may result in tremendous cost.
- (b) To allocate certain components in redundancy in an optimal manner at various stages of a series system so that the overall system reliability is increased.
- (c) To suggest some maintenance policies for an existing system with a view to increase system reliability.

A little concentration on the above discussion reveals that (a) and (b) are useful at the design stage, whereas (c) plays an important role in the evaluation/maintenance problems of the system. More elaborately, the methods developed to meet the requirements of (a) and (b) lose their importance,

once a system is designed and put into operation. Thus, at the latter stage viz., operation stage, one must look for methods that may help in better maintenance of the system. Besides, problems at the maintenance stage are still more important because, sometimes state of the art does not allow a further increase in components' reliability inspite of one's willingness to bear extraordinary amount of money.

There has been a large number of papers in the above directions. [Barlow and Proschan (1965), Bellman and Dreyfus (1958, 1962), Kettle (1962), Proschan and Bray (1965), Fyffe et al. (1968), Hadley (1964), Messigner and Shooman (1970), Jensen (1970), Woodhouse (1972) and Misra (1971a).

Misra (1971a) gave a dynamic programming formulation to determine optimal number of redundant components at each stage in a series system. Lagrangian Multiplier Technique was used to solve the problem. Initially, the problem was formulated under one constraint. However, redundancy allocation under two constraints has also been dealt with.

Bellman and Dreyfus (1962) have discussed the reliability R of a system containing several subsystems, each of which has a redundancy. They maximized the reliability subject to a cost constraint. Kettle (1962) has discussed an approach of least cost allocation with a view to minimize sum of subsystems unreliabilities instead of maximizing system reliability. This was extended by Proschan and Bray (1965) and Barlow and Proschan (1965) to the case of two constraints.

In formulating and solving redundancy allocation problems through dynamic programming with a view to optimize system reliability, number of

constraints causes difficulty because of increased dimensionality. This difficulty has been successfully overcome by employing Lagrangian Multiplier Technique by Bellman and Dreyfus (1958), Fyffe et al. (1968), Hadley (1964), Messigner and Shooman (1970) . Kettle (1962) and Woodhouse (1972) have used the concept of dominating sequences for the purpose.

However, (c) has also attracted attention of some of the workers. Kumar and Lal (1980) studied a 2-unit cold standby redundant system and determined an optimal maintenance policy which maximizes expected profit of the system assuming that there is no discounting. Later on Kumar and Kapoor (1981a) extended the procedure to cover the case of discounted cost structure.

It may be worthwhile to point out that in most of the papers cited above, the methods of analysis are based on dynamic programming formulation. In fact, dynamic programming is a very powerful technique and its spectrum is quite wide in that it enables solution procedures for a fairly large reliability situations. Howard's (1971) policy iteration method is also a version of dynamic programming.

In the present chapter we discuss the following problem for series systems, viz., Optimum cost allocation under a constraint on the cost in the subsequent section.

4.2 OPTIMUM COST ALLOCATION IN A SERIES SYSTEM UNDER A CONSTRAINT ON THE COST

System Model

There is an n -component series system. The problem is to allocate cost to each component with a view to maximize system reliability subject to total cost. It is assumed that there exist a number of components with varying costs and reliability which are capable of performing the same system task. The reliability of each component is assumed to be monotonically increasing function of its cost.

Notation

- $R_n(C_n)$: reliability of component n
 C_n : cost of component n
 R : system reliability
 N : number of components in the system.

System Reliability

Then, the system reliability R is given as

$$R(C_1, C_2, \dots, C_N) = \prod_{n=1}^N R_n(C_n)$$

Now, we obtain a suitable expression for $R_n(C_n)$. Beriphol (1961) has expressed the cost of a component as a function of its reliability :

$$C_n = \frac{K_{1n}}{1 - R_n(C_n)} \exp [-K_{2n} \{1 - R_n(C_n)\}] \quad (4.2.1)$$

where K_{1n} and K_{2n} are constants.

Equation (4.2.1) cannot be solved for $R_n(C_n)$.

Therefore, we devise a function given below in (4.2.2) which leads to approximately the same set of values for reliability and cost as is given by (4.2.1) for $0.9 < R_n(C_n) < 1$.

$$R_n(C_n) = a_n + b_n \log C_n \quad (4.2.2)$$

where $0 < a_n < 1$, $b_n > 0$ and \log is to the base e .

Now the actual problem becomes

$$\begin{aligned} &\text{Maximize} && R(C_1, C_2, \dots, C_N) \\ &C_1, C_2, \dots, C_N \\ &\text{subject to} \\ &\sum_{n=1}^N C_n = C, \quad C_n > 0 \\ &\text{i.e., Maximize} && \left[\prod_{n=1}^N (a_n + b_n \log C_n) \right] \end{aligned}$$

$$C_1, C_2, \dots, C_N$$

subject to

$$\sum_{n=1}^N C_n = C, \quad C_n > 0 \quad (4.2.3)$$

Dynamic Programming Formulation

With a view to maximize system reliability in N stages, define stage j ($j=1, 2, \dots, N$) to consist of j components counted from the last i.e., N to

$(N-j+1)$.

Define

$F(j, C_{N-j+1}, C)$ = Reliability of stage j when cost C_{N-j+1} is allocated to component $N-j+1$, $C-C_{N-j+1}$ is optimally allocated to the remaining $j-1$ components contained in stage j , $j=1, 2, \dots, N$.

$F^*(j, C)$ = Maximum of $F(j, C_{N-j+1}, C)$ with respect to

$$C_{N-j+1}, \quad 0 < C_{N-j+1} < C, \quad j = 1, 2, \dots, N. \quad (4.2.4)$$

Evidently, $F^*(N, C) = \text{Max} [R(C_1, C_2, \dots, C_N)]$

with respect to C_1, C_2, \dots, C_N

subject to

$$C_1 + C_2 + \dots + C_N = C$$

Bellman's Principle of optimality results in the following functional equations :

$$F^*(j, C) = \text{Max} [R_{N-j+1}(C_{N-j+1}) F^*(j-1, C-C_{N-j+1})]$$

with respect to C_{N-j+1} , $0 < C_{N-j+1} < C$, $j = 2, \dots, N$

$$F^*(1, C) = \text{Max} [R_N(C_N)]$$

with respect to C_N , $0 < C_N < C$ (4.2.5)

The optimum values of C_1, C_2, \dots, C_N can be obtained by solving above functional equations.

If standard dynamic programming procedures are followed to solve (4.2.5) the solution is quite voluminous and cumbersome. Therefore, it is

desirable to achieve approximate solutions. We describe below a simple approach to obtain approximate but explicit expression for optimal costs through an illustration.

Illustration

Let

$$N = 3,$$

From (4.2.5)

$$\begin{aligned} F^*(1, C) &= \text{Max } [R_3(C_3)] \text{ with respect to } C_3, 0 < C_3 < C \\ &= \text{Max } [a_3 + b_3 \log C_3] \text{ with respect to } C_3, 0 < C_3 < C \\ &= a_3 + b_3 \log C \end{aligned} \quad (4.2.6a)$$

Evidently,

$$\begin{aligned} F(2, C_2, C) &= R_2(C_2) F^*(1, C-C_2) \\ &= (a_2 + b_2 \log C_2) [a_3 + b_3 \log (C-C_2)] \end{aligned} \quad (4.2.6b)$$

Let

$$K = C_2/C \quad (0 < K < 1)$$

$$g_i = a_i + b_i \log C, \quad i = 1, 2, 3$$

Then

$$F(2, CK, C) = g_2 g_3 + b_3 g_2 \log (1-K) + b_2 g_3 \log K + b_2 b_3 \log K \log (1-K)$$

Neglecting $b_2 b_3 \log K \log (1-K)$ as b_2 and b_3 are small quantities.

$$F(2, CK, C) = g_2 g_3 + b_3 g_2 \log (1-K) + b_2 g_3 \log K = \varphi(K)$$

The maximum value of K denoted by K^* is obtained by differentiating

$\varphi(K)$ with respect to K and equating it to zero i.e., $\frac{d\varphi(K)}{dK} = 0$ which gives

$$K^* = \frac{b_2 g_3}{b_3 g_2 + b_2 g_3}$$

To decide that K^* provides maximum, we further examine the sign of

$$\left. \frac{d^2\varphi(K)}{dK^2} \right|_{K=K^*} \text{ i.e.,}$$

$$\frac{d^2\varphi(K)}{dK^2} = - \left[\frac{b_2 g_3}{K^2} + \frac{b_3 g_2}{(1-K)^2} \right]$$

Thus,

$$\left. \frac{d^2\varphi(K)}{dK^2} \right|_{K=K^*} < 0$$

So K^* maximizes $\varphi(K)$ and hence

$$F^*(2, C) = \varphi(K^*) = F(2, CK^*, C)$$

On similar lines for third stage

$$\begin{aligned} F(3, C_1, C) &= R_1(C_1)F^*(2, C-C_1) \\ &= [a_1 + b_1 \log C_1] [a_2 + b_2 \log (C-C_1) + b_2 \log K^*] . \\ &\quad [a_3 + b_3 \log (C-C_1) + b_3 \log (1-K^*)] \end{aligned}$$

Let

$$L = C_1/C \quad (0 < L < 1)$$

Then

$$\begin{aligned}
F(3, CL, C) &= [a_1 + b_1 \log C + b_1 \log L] [a_2 + b_2 \log C + b_2 \log (1-L) \\
&\quad + b_2 \log K^*] [a_3 + b_3 \log C + b_3 \log (1-L) \\
&\quad + b_3 \log (1-K^*)] \\
&= [h_1 + b_1 \log L] [h_2 + b_2 \log (1-L)] [h_3 + b_3 \log (1-L)]
\end{aligned}$$

where

$$h_1 = g_1$$

$$h_2 = g_2 + b_2 \log K^*$$

$$h_3 = g_3 + b_3 \log (1-K^*).$$

So, on neglecting the terms containing $b_1 b_2$, $b_2 b_3$, $b_1 b_3$ and $b_1 b_2 b_3$ as b_1 , b_2 , b_3 are small quantities.

$$\begin{aligned}
F(3, CL, C) &= h_1 h_2 h_3 + h_1 h_2 b_3 \log (1-L) + h_1 b_2 h_3 \log (1-L) \\
&\quad + b_1 h_2 h_3 \log L \\
&= \theta(L) \text{ (say).}
\end{aligned}$$

The maximum value of L denoted by L^* is obtained by differentiating $\theta(L)$ with respect to L and equating it to zero i.e.,

$$\frac{d\theta(L)}{dL} = 0$$

which gives

$$L^* = \frac{b_1 h_2 h_3}{b_1 h_2 h_3 + b_2 h_1 h_3 + b_3 h_1 h_2}$$

To decide L^* provides maximum, we further examine the sign of

$$\left. \frac{d^2\theta(L)}{dL^2} \right|_{L=L^*} \text{ i.e.,}$$

$$\frac{d^2\theta(L)}{dL^2} = - \left[\frac{b_1 h_2 h_3}{L^2} + \frac{b_2 h_1 h_3 + b_3 h_1 h_2}{(1-L)^2} \right]$$

$$\text{Thus } \left. \frac{d^2\theta(L)}{dL^2} \right|_{L=L^*} < 0$$

So, L^* maximizes $\theta(L)$.

Optimum values of C_1 , C_2 , C_3 denoted by C_1^* , C_2^* , C_3^* , are obtained as follows

$$C_1^* = CL^*, \quad C_2^* = CK^* (1-L^*), \quad C_3^* = C(1-K^*) (1-L^*).$$

Illustration

We now illustrate the results numerically. Let us take $C = 15,00,000$.

Further, assume the following data taken from Beriphol (1961)

R_1	C_1	R_2	C_2	R_3	C_3
.999	1820	.999	1030	.999	6310
.995	100	.995	100	.995	200

Using the above set of values of R_n and C_n ($n=1, 2, 3$), the values of a_n and b_n ($n=1, 2, 3$) are obtained from (4.2.2) as

$$a_1 = .97910, \quad a_2 = .97530, \quad a_3 = .98086$$

$$b_1 = .00138, \quad b_2 = .00171, \quad b_3 = .00116$$

Consequently,

$$K^* = .5937, \quad L^* = .3246$$

and

$$C_1^* = 4,86,900; \quad C_2^* = 6,03,503.67; \quad C_3^* = 4,09,596.33$$

Evidently in this case the maximum attainable reliability of the system due to optimum allocation of cost to each component for a given total cost of 15 lakhs is .99111.

BIBLIOGRAPHY:

1. **Agarwal, M. (1975)** : 'Reliability behaviour of a redundant complex system with standby failure', *Cahiers du Centre d' Etudes de Recherche operationnelle* (Belgium) Vol.17. pp. 1-21.
2. **Aggarwal, K.K. (1976)** : 'Redundancy optimization in general systems', *IEEE Trans. on Rel.*, Vol. R-25, No.5, pp.330-332.
3. **Aggarwal, K.K. (1977)** : 'Minimum cost with specified reliability', *IEEE Trans. on Rel.*, Vol. R-26, No. 3, pp. 166-167.
4. **Aggarwal, K.K., Gupta, J.S. and Misra, K.B. (1975)** : 'A new heuristic criterion for solving a redundancy optimization problem', *IEEE Trans. on Rel.*, Vol. R-24, No.1, pp.86-87.
5. **Aggarwal, K.K. and Gupta, J.S. (1975)** : 'On minimising cost of reliable systems', *IEEE Trans. On Rel.*, Vol. R-24, No.3, pp.205.
6. **Alam, M. and Sarma, V.V.S. (1974)** : 'Optimum maintenance policy for an equipment subject to deterioration and random failure', *IEEE Trans. Syst., Man. Cybern.*, Vol. SMC-4, pp.172-175, March.
7. **Arndt, K. (1977)** : 'Calculation of the reliability of a duplicated system by the method of embedded semi-Markov processes, *Engg., Cyb.*, Vol.15, No.3, pp. 54-63.
8. **Arndt, K. and Franken, P. (1977)** : 'Random point processes applied to availability analysis of redundant system with repair', *IEEE Trans. On Rel.*, Vol. R-26, No.4, pp. 266-269.

9. **Arnold, T.F. (1973)** : 'The concept of coverage and its effect on the reliability model of a repairable system', *IEEE Trans. On Computers*, Vol., C-22, No.3, pp. 251-254.
10. **Banerjee, S.K. and Rajamani, K. (1973)** : 'Optimization of system reliability using a parametric approach', *IEEE Trans. On Rel.*, Vol. R-22, pp. 35-39, April.
11. **Barlow, R.E. and Proschan, F. (1965)** : '*Mathematical Theory of Reliability*', Wiley, New York.
12. **Barlow, R.E. and Proschan, F. (1975)** : '*Statistical Theory of Reliability and Life Testing*', Holt, Rinehart and Winston, New York.
13. **Bellman, R.E. and Dreyfus, S. (1958)** : 'Dynamic programming and reliability of multicomponent devices', *Oper. Res.*, Vol. 6, No. 2 pp. 200-206.
14. **Bellman, R.E. and Dreyfus, S.E. (1962)** : '*Applied Dynamic Programming*', Princeton Univ. Press, Princeton, N.J.
15. **Beraha, D. and Misra, K.B. (1974)** : 'Reliability optimization through random search algorithm', *Micro. and Rel.*, Vol. 13, pp. 295-297, August.
16. **Beriphol, A.M. (1961)** : 'A unique allocation of required component reliability', *Proc. 7th Nat. Symp. Rel. Quality Control*, pp. 189-202, Jan. 9-11.
17. **Birolini, A. (1975)** : 'Some applications of regenerative stochastic processes to reliability theory - Part two : Reliability and availability of 2-item redundant systems', *IEEE Trans. on Rel.*, Vol. R-24, pp. 336-339, December.

18. **Branson, M.H. and Shah, B. (1971)** : 'Reliability analysis of systems comprised of units with arbitrary repair time distributions', *IEEE Trans. On Rel.*, Vol. R-20, pp. 217-223, November.
19. **Butterworth, R. (1972)** : 'System reliability fault testing models', *Oper. Res.*, Vol. 20, No.2, pp. 335-343.
20. **Buzacott, J.A (1971)** : 'Availability of priority standby redundant systems', *IEEE Trans. On Rel.*, Vol. R-20, pp. 60-63.
21. **Chow, D.K. (1972)** : 'Reliability of some modularly redundant systems', *IEEE Trans. On Rel.*, Vol. R-21, pp. 60-70. May.
22. **Chow, D.K. (1973)** : 'Reliability of some redundant systems with repair', *IEEE Trans. on Rel.*, Vol. R-22, pp. 223-228. October.
23. **Chow, D.K. (1975)** : 'Availability of some repairable computer systems', *IEEE Trans., on Rel.*, Vol. R-24, No. 3, pp. 221-223.
24. **Cleroux, R. and Hanscom, M. (1974)** : 'Age replacement with adjustment and depreciation costs and interest charges', *Technometrics*, Vol.16, pp. 235-239, May.
25. **Diveroli, J.C. (1974)** : 'Optimal continuous policies for repair and replacement', *Opl. Res. Quart.*, Vol. 25, pp. 89-97.
26. **Federowicz, A.J. and Mazumdar, M. (1968)** : 'Use of geometric programming to maximize reliability achieved by redundancy', *Oper. Res.*, Vol. 16, No. 5 pp. 948-954.
27. **Fox, B. (1966)** : 'Age replacement with discounting', *Oper. Res.*, Vol. 14, pp. 533-537.

28. Fyffe, D.E., Hines, W.W. and Lee, N.K. (1968) : 'System Reliability allocation and a computational algorithm', *IEEE Trans. on Rel.*, Vol. R-17, No.2, pp. 64-69.
29. Garg, R.C. (1963a) : 'Dependability of a complex system having two types of components', *IEEE Trans. on Rel.*, Vol. R-12, pp. 11-16, September.
30. Garg, R.C. (1963b) : 'Dependability of a complex system with general waiting time distributions', *IEEE Trans. on Rel.*, Vol. R-12, pp. 17-21.
31. Garg, R.C. and Kumar, A. (1977) : 'A complex system with two types of failure and repair', *IEEE Trans. on Rel.*, Vol. R-26, pp. 299-300, October.
32. Gaver, D.P. (1963) : 'Time to failure and availability of paralleled systems with repair', *IEEE Trans. on Rel.*, Vol. R-12, pp. 30-38, June.
33. Gaver, D.P. (1964) : 'Failure time for a redundant repairable system of two dissimilar elements', *IEEE Trans. on Rel.*, Vol. R-13, pp. 14-17, March.
34. Ghare, P.M. and Taylor, R.E. (1969) : 'Optimal redundancy for reliability in series systems', *Oper. Res.*, Vol. 17, No. 5, pp. 838-847.
35. Glasser, G.J. (1967) : 'The age replacement problem', *Technometrics*, Vol. 9, pp. 83-91.
36. Gnedenko, B.V., Belyayer, Y.K. and Solovyev, A.D. (1969) : '*Mathematical Methods of Reliability Theory*', Academic Press, New York.

37. **Gopalan, M.N. (1975)** : 'Availability and reliability of 1-server 2-unit system with imperfect switch', *IEEE Trans. on Rel.*, Vol. R-24 pp. 218-219, August.
38. **Gopalan, M.N. and D'Souza, C.A. (1975)** : 'Probabilistic analysis of a two-unit system with a single service facility for preventive maintenance and repair, *Oper. Res.*, Vol. 23, pp. 173-177, Jan-Feb.
39. **Gopalan, M.N. and Marathe, K.Y. (1978)** : 'Availability of 1-server 2-dissimilar unit system with slow switch', *IEEE Trans. on Rel.*, Vol. R-27, pp. 230-231, August.
40. **Gopalan, M.N. and Natesan, J. (1982)** : 'Cost benefit analysis of a 1-server 2-unit system subject to different repair strategies', *Micro. and Rel.*, Vol. 22, No. 3, pp. 393-397.
41. **Gopalan, M.N. and Saxena, A.R. (1977)** : 'Analysis of an n-unit system with 1-repair facility which has down-time', *IEEE Trans. on Rel.*, Vol. R-26, p. 297, October.
42. **Gross, A.J. (1970)** : 'Minimization of misclassification of component failures in a two-component system', *IEEE Trans. on Rel.*, Vol. R-19, No.3, pp. 120-122.
43. **Gupta, M.C. (1974)** : 'Availability model with up time and down time costs', *Cahiers du Centre d'Etudes de Recherche Operationnelle*, Vol.16, No. 3, pp. 231-238.
44. **Gupta, M.C. and Kumar, A. (1981)** : 'On profit evaluation in a two-unit maintained redundant system', *Micro. and Rel.*, Vol. 21, No. 4, pp. 605-608.

45. Gupta, M.C. and Kumar, A.(1983) : 'On profit considerations of a maintained system with minor repair', *Micro. and Rel.*, Vol. 23, No. 3, pp. 437-439.
46. Gupta, M.C. and Kumar, A.(1983) : 'On profit evaluation in a modular system with two types of failurers', *Micro. and Rel.*, Vol. 23, No.5, pp. 823-825.
47. Gupta, M.C. and Kumar, A.(1983) : 'Optimal failure classification in an n-unit series system', *Int. Jr. of Systems Science*, Vol. 14, No. 5, pp. 513-518.
48. Gupta, P.P. (1973a) : 'Complex system reliability with general repair time distribution under preemptive repeat repair discipline', *Micro. and Rel.*, Vol. 12, pp. 145-150, April.
49. Gupta, P.P. (1973b) : 'Complex system reliability with general repair time distributions under preemptive resume repair discipline', *Micro. and Rel.*, Vol.12, pp. 351-356, October.
50. Hadley, G. (1964) : '*Nonlinear and Dynamic Programming*', Addison-Wesley, Reading, Mass.
51. Hastings, N.A.J. (1969) : 'The repair limit replacement method', *Opl. Res. Quart.*, Vol. 20, pp. 337-349.
52. Hiller, F.S. and Liberman, G.J. (1967) : '*Introduction to Operations Research*', Holden-Day, San Francisco, Calif.
53. Howard, R.A. (1960) : '*Dynamic Programming and Markov processes*', MIT, Technology Press, Cambridge, Mass.

54. **Howard, R.A. (1971)** : '*Dynamic Probabilistic Systems*', Vol. II, Wiley, New York.
55. **Hwang, C.L., Lai, K.C., Tillman, F.A. and Fan, L.T. (1975)** : 'Optimization of system reliability by the sequential unconstrained minimization technique', *IEEE Trans. on Rel.*, Vol. R-24, No.2, pp. 133-135.
56. **Hwang, C.L., Tillman, F.A. and Kuo, W.(1979)** : 'Reliability optimization by generalized Lagrangian function and reduced-gradient methods', *IEEE Trans. on Rel.*, Vol. R-28, No.4, pp. 316-319.
57. **Hyun, K.N. (1975)** : 'Reliability optimization by 0-1 programming for a system with several failure modes', *IEEE Trans. on Rel.*, Vol. R-24, pp. 206-210, August.
58. **Jensen, P.A. (1970)** : 'Optimization of series-parallel-series networks', *Oper. Res.*, Vol. 18, No.3, pp. 471-482.
59. **Kaio, N. and Osaki, S. (1981)** : 'Optimum repair limit policies with a cost constraint', *Micro. and Rel.*, Vol. 21, pp. 597-599.
60. **Kapil, D.V.S. and Sinha, S.M. (1981)** : 'Repair limit suspension policies for a 2-unit redundant system with 2-phase repairs', *IEEE Trans. on Rel.*, Vol. R-30, pp. 90-91, April.
61. **Kapur, P.K. and Kapoor, K.R. (1975a)** : 'A 2-unit warm standby redundant system with delay and one repair facility', *IEEE Trans. on Rel.*, Vol. R-24. p. 275, October.
62. **Kapur, P.K. and Kapoor, K.R. (1975b)** : 'Intermittent complex systems', *IEEE Trans. on Rel.*, Vol. R-24, pp. 276-277, October.

63. **Kapur, P.K. and Kapoor, K.R. (1978)** : 'Stochastic behaviour of some 2-unit redundant systems', *IEEE Trans. on Rel.*, Vol. R-27, pp. 382-385, December.
64. **Kettle, J.D.(1962)** : 'Least cost allocations of reliability investment', *Oper. Res.* Vol. 10, No. 2, pp. 249-265.
65. **Khalil, Z.S. (1977a)** : 'On the reliability of a two-unit redundant system with random switch over time and two types of repair', *Micro. and Rel.*, Vol. 16, pp. 159-160.
66. **Khalil, Z.S. (1977b)** : 'On the reliability of a 3-unit redundant system with two types of repair', *Micro. and Rel.*, Vol. 16, pp. 689-690.
67. **Khalil, Z.S. and Bougas, I. (1975)** : 'On standby redundancy with delayed repair', *IEEE Trans. on Rel.*, Vol. R-24, pp. 286-287, October.
68. **Kodama, M. and Deguchi, H. (1974)** : 'Reliability considerations for a 2-unit redundant system with Erlang failure and general repair distributions', *IEEE Trans. on Rel.*, Vol. R-23, pp. 75-81, June.
69. **Kontoleon, J.M. (1977)** : 'Optimum redundancy of repairable modules', *IEEE Trans. on Rel.*, Vol. R-26, No. 4, pp. 277-279.
70. **Kulshrestha, D.K. (1968)** : 'Reliability of a parallel redundant complex system', *Oper. Res.*, Vol. 16, pp. 28-35, Jan-Feb.
71. **Kulshrestha, D.K. (1970)** : 'Reliability of a repairable multi-component system with redundancy in parallel', *IEEE Trans. on Rel.*, Vol. R-19, pp. 50-52, May.

72. **Kulshrestha, D.K. (1972)** : 'Application of discrete transforms in reliability of parallel redundant system', *Metrika*, Vol. 18, pp. 145-153.
73. **Kulshrestha, D.K. and Gupta, M.C. (1973)** : 'Use of dynamic programming for reliability engineers', *IEEE Trans. on Rel.*, Vol. R-22, No. 4, pp. 240-241.
74. **Kumar, A. (1975)** : 'Optimum detection of failures in an n-component system', *ZAMM*, Vol. 55, No. 11, pp. 679-680.
75. **Kumar, A. (1976)** : 'Steady-state profit in a 2-unit standby system', *IEEE Trans. on Rel.*, Vol. R-25, pp. 105-108, June.
76. **Kumar, A. (1977a)** : 'Behaviour of steady-state profit in a 2-unit cold standby redundant system', *ZAMM*, Vol. 57, pp. 124-127, February.
77. **Kumar, A. (1977b)** : 'Stochastic behaviour of a 2-unit standby redundant system with two switching failure modes', *Math. Operf. und Statisk Ser. Optimization*, Vol. 8, pp. 571-580, October.
78. **Kumar, A. (1977c)** : 'Steady-state profit in several 1-out-of - 2 : G systems', *IEEE Trans. on Rel.*, Vol. R-26, pp. 366-369, December.
79. **Kumar, A. (1977d)** : 'On expected profit in a 2-unit standby system with switch failure', *Int. Jr. of Systems Science*, Vol. 8, pp. 145-151, February.
80. **Kumar, A. and Agarwal, M. (1978)** : 'Analysis of a two-unit standby redundant system with two types of failures', *IEEE Trans. on Rel.*, Vol. R-27, pp. 301-302, October.

81. **Kumar, A. and Agarwal,, M. (1980)** : 'A review of standby redundant systems', *IEEE Trans. on Rel.*, Vol. **R-29**, pp. 290-304, October.
82. **Kumar, A. and Gupta, M.C.(1984)** : 'Optimal allocation of cost to detectors in a two-unit series system', *Micro. and Rel.*, Vol. **24**, pp. 633-636.
83. **Kumar, A. and Kapoor, V.B. (1979)** : 'Analysis of a two-unit maintained series system with imperfect detection of failures', *Micro. and Rel.*, Vol. **19**, pp. 329-331.
84. **Kumar, A. and Kapoor, V.B. (1981a)** : 'Optimal Maintenance of A two-unit standby redundant system with a generalized cost structure', *Micro. and Rel.*, Vol. **21**, No. **1**, pp. 117-120.
85. **Kumar, A. and Kapoor, V.B. (1981b)** : 'Optimal classification of failures in a two-unit series system', *Int. Jr. of Systems Science*, Vol. **12**, No. **1**, pp. 127-132.
86. **Kumar, A., Kapoor,V.B. and Aggarwal, S.P. (1981)** : 'Stochastic behaviour of a two-unit redundant system with switchover time', *Micro. and Rel.*, Vol. **21**, pp. 717-725.
87. **Kumar, A., Kapoor, V.B. and Gupta, M.C.(1983)** : 'On Optimal maintenance of a series system', *Micro. and Rel.*, Vol. **23**, pp. 827-831.
88. **Kumar, A. and Lal, R. (1979)** : 'Stochastic behaviour of a 2-unit standby system with contact failure and intermittently available repair facility', *Int. Jr. of Systems Science*, Vol. **10**, pp. 589-603.
89. **Kumar, A. and Lal, R.(1980)** : 'Profit optimization in a two-unit maintained standby redundant system', *ZAMM*, Vol. **60**, pp. 725-732.

90. Kuo, W., Hwang, C.L. and Tillman, F.A. (1978) : 'A note on heuristic methods in optimal system reliability', *IEEE Trans. on Rel.*, Vol. R-27, No. 5 pp. 320-324.
91. Lal, R., Kumar, A. and Joshi, D.D. (1980) : '2-unit standby system with fault analysis', *IEEE Trans. on Rel.*, Vol. R-29, No. 5, pp. 431-432.
92. Lambert, B.K., Walvekar, A.G. and Hirmas, J.P. (1971) : 'Optimal redundancy and availability allocation in multistage system', *IEEE Trans. on Rel.*, Vol. R-20, No. 3, pp. 182-185.
93. Linton, D.G. (1976) : 'Some advances in the analysis of two-unit parallel redundant systems', *Micro. and Rel.*, Vol. 15, pp. 39-46.
94. Linton, D.G. and Braswell, R.N. (1973) : 'Laplace transforms for the two-unit cold-standby redundant systems', *IEEE Trans. on Rel.*, Vol. R-22, pp. 105-108, June.
95. Linton, D.G. and Saw, J.G. (1974) : 'Reliability analysis of the k-out-of-n: F system', *IEEE Trans. on Rel.*, Vol. R-23, pp. 97-103, June.
96. Luus, R. (1975) : 'Optimization of system reliability by a new nonlinear integer programming procedure', *IEEE Trans. on Rel.*, Vol. R-24, No. 1, pp. 14-16.
97. Mcleavey, D.W. (1974) : 'Numerical Investigation of optimal parallel redundancy in series systems', *Oper. Res.*, Vol. 22, No. 5, pp.1110-1117.
98. Mcleavey, D.W. and Mcleavey, J.A. (1976) : 'Optimization of system reliability by branch-and-bound', *IEEE Trans. on Rel.*, Vol. R-25, No. 5, pp. 327-329.

99. **Messinger, M. and Shooman, M.L. (1970)** : 'Techniques for optimum spares allocation : A tutorial review', *IEEE Trans. on Rel.*, Vol. **R-19**, No. 4, pp. 156-166.
100. **Mine, H. and Kawai, H. (1974)** : 'An optimal maintenance policy for a 2-unit parallel system with degraded states', *IEEE Trans. on Rel.*, Vol. **R-23**, pp. 81-86, June.
101. **Mine, H. and Kawai, H. (1975)** : 'An optimal Inspection and replacement policy', *IEEE Trans. on Rel.*, Vol. **R-24**, pp. 305-309, December.
102. **Mine, H. and Nakagawa, T. (1978)** : 'Age replacement model with mixed failure times', *IEEE Trans. on Rel.*, Vol. **R-27**, pp. 173, June.
103. **Misra, K.B. (1971a)** : 'Dynamic programming formulation of the redundancy allocation problem', *Int. Jr. Math. Edu. Sci. Techno.*, Vol. 2, pp. 207-215, July-August.
104. **Misra, K.B. (1971b)** : 'A method of solving redundancy optimization problem', *IEEE Trans. on Rel.*, Vol. **R-20**, pp. 117-120, August.
105. **Misra, K.B. (1972)** : 'A simple approach for constrained redundancy optimization problem', *IEEE Trans. on Rel.*, Vol. **R-21**, pp. 30-34, February.
106. **Misra, K.B. and Ljubojevic, M.D. (1973)** : 'Optimal Reliability design of a system :- a new look', *IEEE Trans. on Rel.*, Vol. **R-22**, pp. 255-258.
107. **Misra, K.B. and Sharma, J. (1973)** : 'A new geometric programming formulation for a reliability problem', *Int. Jr. Qual. Cont.*, Vol. 18, No. 3, pp. 497-503.

108. Muth, E.J. (1966) : 'Reliability of a system having standby spare plus multiple-repair capability', *IEEE Trans. on Rel.*, Vol. R-15, pp. 76-81, August.
109. Muth, E.J. (1977) : 'An optimal decision rule for repair vs. replacement', *IEEE Trans. On Rel.*, Vol. R-26, No. 3, pp. 179-181, August.
110. Nakagawa, T. (1977a) : 'A 2-unit repairable redundant system with switching failure', *IEEE Trans. on Rel.*, Vol. R-26, No. 2, pp. 128-130.
111. Nakagawa, T. (1977b) : 'Optimum preventive maintenance policies for repairable systems', *IEEE Trans. on Rel.*, Vol. R-26, pp. 168-173, August.
112. Nakagawa, T. (1980) : 'Replacement policies for a unit with random and wearout failures', *IEEE Trans. on Rel.*, Vol. R-29, pp. 342-344, October.
113. Nakagawa, T. and Osaki, S. (1974a) : 'Stochastic behaviour of a two-unit standby redundant system', *INFOR*, Vol. 12, pp. 60-70, February.
114. Nakagawa, T. and Osaki, S. (1974b) : 'Stochastic behaviour of a two-dissimilar unit standby redundant system with repair maintenance', *Micro. and Rel.*, Vol. 13, pp. 143-148, April.
115. Nakagawa, T. and Osaki, S. (1974c) : 'Optimum replacement policies with delay', *J. Appl. Prob.*, Vol.11, pp. 102-110.
116. Nakagawa, T. and Osaki, S. (1974d) : 'The optimum repair limit replacement policies', *Opl. Res. Quart.*, Vol. 25, No. 2, pp. 311-317.
117. Nakagawa, T. and Osaki, S. (1975) : 'Stochastic behaviour of 2-unit standby redundant system with imperfect switchover', *IEEE Trans. on Rel.*, Vol. R-24, pp. 143-146, June.

118. Nakagawa, T. and Osaki, S.(1976a) : 'Reliability analysis of a one-unit system with unrepairable spare units and its optimization applications', *Opl. Res. Quart.*, Vol. 27, No. 1(i), pp. 101-110.
119. Nakagawa, T. and Osaki, S. (1976b) : 'A summary of optimum preventive maintenance policies for a 2-unit standby redundant system', *Zeitschrift fur Oper. Res.*, Vol. 22, pp. 171-187.
120. Nakagawa, T., Sawa, Y. and Sazuki, Y. (1976) : 'Reliability analysis of intermittently used system when failures are detected only during a usage period', *Micro, and Rel.*, Vol. 15, pp. 35-38.
121. Nakagawa, Y. and Nakashima, K. (1977) : 'A heuristic method for determining optimal reliability allocation,' *IEEE Trans. on Rel.*, Vol. R-26, No. 3, pp. 156-161.
122. Nielsen, D.S. and Runge, B. (1974) : 'Unreliability of a standby system with repair and imperfect switching', *IEEE Trans. on Rel.*, Vol. R-23, pp. 17-24, April.
123. Okumoto, K. and Osaki, S. (1977) : 'Optimum policies for a standby system with preventive maintenance', *Opl. Res. Quart.*, Vol. 28, No. 2 (ii), pp. 415-423.
124. Osaki, S.(1970a) : 'System reliability analysis by Markov renewal process', *J. Oper. Res. Soc. Japan*, Vol. 12, pp. 127-188, May.
125. Osaki, S. (1970b) : 'Renewal theoretic aspects of two-unit redundant systems', *IEEE Trans. on Rel.*, Vol. R-19, pp. 105-110, August.
126. Osaki, S. (1972a) : 'On a two-unit standby redundant system with imperfect switchover', *IEEE Trans. on Rel.*, Vol. R-21, pp. 20-24, February.

127. **Osaki, S. (1972b)** : 'Reliability analysis of a two-unit standby redundant system with preventive maintenance', *IEEE Trans. on Rel.*, Vol. R-21, pp. 24-29.
128. **Osaki, S. and Asakura, T. (1970)** : 'A two-unit standby redundant system with repair and preventive maintenance', *J. Appl. Prob.*, Vol. 7, pp. 641-648, December.
129. **Osaki, S. and Nakagawa, T. (1976)** : 'Bibliography for reliability and availability of stochastic system', *IEEE Trans. on Rel.*, Vol. R-25, pp. 284-287, October.
130. **Osaki, S. and Okumoto, K. (1977)** : 'Repair limit suspension policies for a two-unit standby redundant system with two phase repairs', *Micro. and Rel.*, Vol. 16, pp. 41-45.
131. **Osaki, S. and Nishio, T. (1980)** : 'Reliability Evaluation of Some Fault Tolerant Computer Architectures' Vol. 97, Springer-Verlag, New York.
132. **Pal, M.N. and Bhattacharjee, M.C. (1978)** : 'Redundancy optimization through simplex pattern search', *IEEE Tras. on Rel.*, Vol. R-27, No. 2, pp. 120-123.
133. **Park, K.S. (1982)** : 'Optimal diagnostic procedure for failures in a series system', *Int. Jr. of Systems Science*, Vol. 13, No. 4, pp. 409-412.
134. **Prakash, S. (1973)** : 'Analytic study of a standby redundant equipment with switching and shelf life failures', *Micro. and Rel.*, Vol. 12, pp. 329-335, October.
135. **Proctor, C.L. and Singh, B. (1976)** : 'A repairable 3-state device', *IEEE Trans. on Rel.*, Vol. R-25, pp. 210-211, August.

136. **Proschan, F. (1964)** : '*Statistical Theory of Reliability*', (ed). **M. Zelan** Madison, The University of Wisconsin Press, pp. 50-70.
137. **Proschan, F. and Bray, T.A. (1965)** : 'Optimum redundancy under multiple constraints', *Oper. Res.*, Vol. 13, No.5, pp. 800-814.
138. **Rau, J.G. (1970)** : '*Optimization and Probability in Systems Engineering*', Van Nostrand Reinhold Co., New York.
139. **Sandler, G.H. (1963)** : '*System Reliability Engineering*', Englewood Cliffs, N.J., Prentice-Hall.
140. **Scheaffer, R.L. (1971)** : 'Optimum age replacement policies with an increasing cost factor', *Technometrics*, Vol. 13, pp. 139-144.
141. **Sharma, C.M. (1974)** : 'Time dependent solution of a complex standby redundant system under preemptive repeat repair discipline', *IEEE Trans. on Rel.*, Vol. R-23, pp. 283-285, October.
142. **Sharma, J. and Venkateswaran, K.V. (1971)** : 'A direct method for maximizing the system reliability', *IEEE Trans. on Rel.*, Vol. R-20, No. 4, pp. 256-259.
143. **Shershin, A.C. (1970)** : 'Mathematical optimization techniques for the simultaneous apportionments of reliability and maintainability', *Oper. Res.*, Vol. 18, pp. 95-106.
144. **Shetty, H.V.K. and Sengupta, D.P. (1975)** : 'Reliability optimization using SLUMT', *IEEE Trans. on Rel.*, Vol. R-24, No. 1, pp. 80-82.

145. Srinivasan, S.K. and Gopalan, M.N. (1973a) : 'Probabilistic analysis of a two-unit cold standby system with a single repair facility', *IEEE Trans. on Rel.*, Vol. R-22, pp. 250-254, December.
146. Srinivasan, S.K. and Gopalan, M.N. (1973b) : 'Probabilistic analysis of a two-unit system with a warm standby and a single repair facility', *Oper. Res.*, Vol. 21, pp. 748-754, May-June.
147. Srinivasan, S.K. and Subramanian, R. (1980) : '*Probabilistic Analysis of Redundant Systems*', Springer-Verlag, Berlin.
148. Srinivasan, S.K., Subramanian, R. and Ramesh, K.S. (1971) : 'Mixing of two renewal processes and its applications to reliability theory', *IEEE Trans. on Rel.*, Vol. R-20, pp. 51-55, May.
149. Srinivasan, V.S. (1966) : 'The effect of standby redundancy in system's failure with repair maintenance', *Oper. Res.*, Vol. 14, pp. 1024-1036, Nov.-Dec.
150. Srinivasan, V.S. (1968) : 'A standby redundant model with non-instantaneous switchover', *IEEE Trans. on Rel.*, Vol. R-17, pp. 175-178.
151. Subramanian, R. (1978) : 'Availability of a 2-unit system with preventive maintenance and one-repair facility', *IEEE Trans. on Rel.*, Vol. R-27, pp. 171-173, June.
152. Subramanian, R., Venkatakrishnan, K.S. and Kistner, K.P. (1976) : 'Reliability of a repairable system with standby failure', *Oper. Res.* Vol. 24, No. 1.

153. Takami, I., Imagaki, T., Sakino, E. and Inoue, K. (1978) : 'Optimal allocation of fault detectors', *IEEE Trans. on Rel.*, Vol. R-27, No. 5, pp. 360-362.
154. Tillman, F.A., Hwang, C.L., Fan, L.T. and Lai, K.C. (1970) : 'Optimal reliability of complex system', *IEEE Trans. on Rel.*, Vol. R-19, pp. 95-100, August.
155. Tillman, F.A., Hwang, C.L. and Kuo, W. (1977a) : 'Optimization techniques for system reliability with redundancy - A review', *IEEE Trans. on Rel.*, Vol. R-26, No. 3, pp. 148-155.
156. Tillman, F.A., Hwang, C.L. and Kuo, W. (1977b) : 'Determining component reliability and redundancy for optimum system reliability', *IEEE Trans. on Rel.*, Vol. R-26, No. 3, pp. 162-166.
157. Tillman, F.A., Hwang, C.L. and Kuo, W. (1981) : '*Optimization of System Reliability*', Marcel Dekker, Inc., New York.
158. Tillman, F.A., and Littschwager, J.M. (1967) : 'Integer programming formulation of constrained reliability problems', *Manag. Sci.*, Vol. 13, pp. 887-899, July.
159. Varma, G.K. (1973) : 'Operational behaviour of a complex system with two out of M failed components', *Micro. and Rel.*, Vol. 12, pp. 107-110, April.
160. Woodhouse, C.F. (1972) : 'Optimum redundancy allocation by dynamic programming', *IEEE Trans. on Rel.*, Vol. R-21, pp. 60-62, February.